A Simple Introduction to Task-Based Models of Technical Change

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Task-based models of production allow for a rich and insightful analysis of technical change. Here, we provide a simple introduction to the analysis of technical change using a task-based model. Our exposition builds on earlier work by Zeira (1998), Acemoglu and Autor (2011), Acemoglu and Restrepo (2018), Aghion, Jones and Jones (2019), Jones and Liu (2024), and Jones and Tonetti (2026), among others. We discuss how different forms of technical change—one being automation of tasks, another being improved productivity of capital at tasks already performed by capital—affect output, productivity, wages, and the labor share of income. We highlight the importance of the elasticity of substitution across tasks, and, in particular, whether tasks are gross complements or gross substitutes in production.

We use the model to develop five main ideas. First, when tasks are gross complements, output is limited by weak links: even if capital is arbitrarily productive at certain tasks, aggregate output will be finite. Second, technical change contributes to growth through two conceptually distinct channels: machines improving at the tasks they already perform and machines taking over new tasks. Third, when tasks are gross complements, these two channels push the labor share in opposite directions: automation lowers the labor share, while improvements in capital productivity for tasks already performed by capital raises the labor share. Fourth, automation can lower wages. Fifth, if automation and improvements in capital productivity are balanced, the labor share can remain stable.

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1 The Environment

Consider an economy populated by a representative household that supplies labor L_t inelastically, owns a fixed amount of capital K_t that it rents to firms, and derives utility from consuming a final good C_t. The final good is produced by a representative firm. The production process for the final good involves completing a unit measure of tasks. We index tasks by ν and denote the output of task ν by $Q_t(\nu)$. The output of the final good is then given by the following function of the tasks completed:

$$Q_t = \left(\int_0^1 Q_t(\nu)^{\frac{\sigma-1}{\sigma}} \, d\nu \right)^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

where $\sigma > 0$ is the elasticity of substitution across tasks.

We will see below that the elasticity of substitution σ is a key parameter. When $\sigma < 1$, tasks are said to be gross complements, whereas, when $\sigma > 1$, they are said to be gross substitutes. This distinction will be crucial for the results we develop below. When tasks are gross complements, no task is dispensable: every task is essential for production and aggregate output is held back by the tasks that the economy performs least well. This has important implications for how the labor share of income responds to technical change.

The production function for a particular task ν is

$$Q_t(\nu) = \psi_{kt}(\nu) K_t(\nu) + \psi_{\ell t}(\nu) L_t(\nu), \tag{2}$$

where $K_t(\nu)$ denotes the amount of capital used in performing task ν, $L_t(\nu)$ denotes the amount of labor used in performing task ν, and $\psi_{kt}(\nu)$ and $\psi_{\ell t}(\nu)$ denote the task-specific productivity of capital and labor, respectively. Note that we have assumed that capital and labor are perfect substitutes in the production of each task ν. Hence, in this model, factors are perfect substitutes within tasks whereas tasks are imperfect substitutes (potentially gross complements) in production of the final good.

The economy's resource constraints are

$$Q_t = C_t, \quad \int_0^1 K_t(\nu) \, d\nu = K_t \quad \text{and} \quad \int_0^1 L_t(\nu) \, d\nu = L_t.$$

Note that capital is assumed to be produced ex nihilo for simplicity. See, among others, Acemoglu and Restrepo (2022) and Restrepo (2024) for a generalized version where capital is an endogenous intermediate input with a task-specific cost.

The production function (1) is symmetric with respect to the tasks. We can therefore order the tasks however we want. We choose the following ordering:

Assumption 1. *(Relative Productivities) We assume that*

$$\frac{\psi_{kt}(\nu)}{\psi_{\ell t}(\nu)}$$

is decreasing in ν so that labor has a comparative advantage at high-index tasks.

The basic model described above can be given a number of alternative interpretations. For example, the tasks could be interpreted as intermediate goods or even as sectors. Also, richer frameworks would involve additional nested layers. Aggregate output might be a constant elasticity of substitution (CES) function of output from different sectors, which might be a CES function of output of individual goods, which might in turn be produced in CES fashion by a continuum of jobs, each of which may consist of a continuum of tasks. We focus on a simple setting with only two layers with different degrees of substitution.

2 Firm Behavior

The representative firm faces two problems that can be analyzed separately. First, it must choose how much of each task to perform. Second, it must choose how to perform each task (i.e., with capital, with labor, or with some combination of the two). We analyze these two problems in turn starting with the first one.

2.1 Final Good Production

The representative firm chooses how much of each task to perform by solving the following profit maximization problem:

$$\max_{[Q_t(\nu)]_{\nu\in[0,1]}} Q_t - \int_0^1 P_t(\nu)\, Q_t(\nu)\, d\nu,$$

where $P_t(\nu)$ denotes the price of task ν, and we assumed that the final good is the numeraire so that its price is normalized to 1. The first-order condition of the firm's problem is then given by

$$Q_t(\nu) = P_t(\nu)^{-\sigma} Q_t. \tag{3}$$

This is the firm's demand curve for task ν. It is downward sloping in the price of task ν with an elasticity of demand equal to $-\sigma$.

It is instructive to divide the demand for one task ν by the demand for another task ν'. This yields

$$\frac{P_t(\nu)}{P_t(\nu')} = \left[\frac{Q_t(\nu)}{Q_t(\nu')}\right]^{-\frac{1}{\sigma}}. \tag{4}$$

If tasks are gross complements ($\sigma < 1$), this equation implies that relative prices change more than proportionately when relative quantities change. For example, a one percent increase in the relative quantity produced of task ν will be associated with a more than one percent fall in the relative price of task ν. This is a manifestation of Baumol (1967)'s cost disease: in an economy with two sectors—one rapid-growing and one slow-growing—the slow-growing sector becomes increasingly important over time. As the rapid-growing sector becomes relatively more abundant, its relative price falls more than proportionately. This means that the slow-growing sector absorbs an increasing share of nominal output and employment. The same force operates across tasks in our framework when tasks are gross complements: as some tasks become relatively abundant, their relative prices collapse and the scarce tasks come to dominate the cost of producing the final good. This complementarity-driven force is essential for the weak-links logic that we develop in Section 3.

The ideal-price index for the final good in our economy is given by

$$1 = \left[\int_0^1 P_t(\nu)^{1-\sigma} d\nu\right]^{\frac{1}{1-\sigma}}.$$

2.2 The Production of Tasks

The task production function—equation (2)—implies that the cost of producing one unit of task ν with labor is $\frac{W_t}{\psi_{\ell t}(\nu)}$, while the cost of producing one unit of this task with capital is $\frac{R_t}{\psi_{kt}(\nu)}$. Since the task production function is linear in labor and capital, these costs do not vary with the quantity of labor and capital used for the production of the task.

The firm will choose the cheapest factor to produce each task. If $\frac{W_t}{\psi_{\ell t}(\nu)} < \frac{R_t}{\psi_{kt}(\nu)}$, task ν will be produced with labor, while if $\frac{W_t}{\psi_{\ell t}(\nu)} > \frac{R_t}{\psi_{kt}(\nu)}$, it will be produced with capital. (If

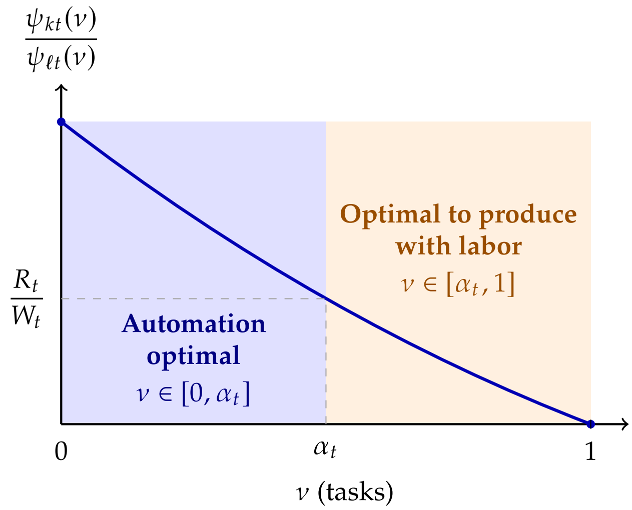


Figure 1. Smooth Automation.

Note: The figure shows the case where $\frac{\psi_{kt}(\nu)}{\psi_{\ell t}(\nu)}$ is continuous and declines smoothly in ν, reaching zero at $\nu = 1$, with automation optimal below the cutoff α_t. The cutoff is determined in equilibrium by the indifference condition $\frac{R_t}{\psi_{kt}(\alpha_t)} = \frac{W_t}{\psi_{\ell t}(\alpha_t)}$, which the dashed lines illustrate: the curve crosses the relative factor price R_t/W_t at α_t, and automation is optimal for tasks with $\frac{\psi_{kt}(\nu)}{\psi_{\ell t}(\nu)} > \frac{R_t}{W_t}$.

$\frac{W_t}{\psi_{\ell t}(\nu)} = \frac{R_t}{\psi_{kt}(\nu)}$, the firm is indifferent.)

This implies that the marginal cost of producing task ν is

$$MC_t(\nu) = \min\left\{\frac{W_t}{\psi_{\ell t}(\nu)}, \frac{R_t}{\psi_{kt}(\nu)}\right\}.$$

Since we assume that the market for producing tasks is perfectly competitive, the price of task ν will equal its marginal cost:

$$P_t(\nu) = MC_t(\nu) = \min\left\{\frac{W_t}{\psi_{\ell t}(\nu)}, \frac{R_t}{\psi_{kt}(\nu)}\right\}. \tag{5}$$

Assumption 1 says that the relative productivity of capital $\frac{\psi_{kt}(\nu)}{\psi_{\ell t}(\nu)}$ is decreasing in ν. This means that the cost of producing goods with capital relative to labor is increasing in ν. Figure 1 plots an example where $\frac{\psi_{kt}(\nu)}{\psi_{\ell t}(\nu)}$ declines smoothly across tasks. In this case, there will be a cutoff α_t with all tasks to the left of α_t (i.e., $\nu \in [0, \alpha_t]$) produced by capital and all tasks to the right of α_t (i.e., $\nu \in [\alpha_t, 1]$) produced by labor. The location of this automation cutoff α_t is determined in equilibrium by the indifference condition

$$\frac{R_t}{\psi_{kt}(\alpha_t)} = \frac{W_t}{\psi_{\ell t}(\alpha_t)}.$$

We characterize the equilibrium of this smooth-automation case in Appendix A.

Here, we focus on a simple case in which $\psi_{kt}(\nu)$ and $\psi_{\ell t}(\nu)$ are uniform across tasks except that only tasks $\nu \in [0, \alpha_t]$ can be produced by capital. Formally, we make the following assumption.

Assumption 2. *(Uniform Productivities) We assume that $\psi_{kt}(\nu) = \psi_{kt}$ for $\nu \in [0, \alpha_t]$ while $\psi_{kt}(\nu) = 0$ for $\nu \in [\alpha_t, 1]$ and $\psi_{\ell t}(\nu) = \psi_{\ell t}$ for $\nu \in [0, 1]$.*

This assumption allows us to cleanly separate two aspects of technical change: automation—i.e., shifts in α_t—and increases in the productivity of capital in tasks already performed by capital—i.e., increases in ψ_{kt}. These two forms of technical change turn out to have very different implications for labor. A shift in α_t can be thought of as a jump in $\psi_{kt}(\nu)$ for task α_t from zero to ψ_{kt}. In some cases, this may be thought of as the invention of a new type of machine (e.g., the steam engine or electric motor).

The sharp distinction we make between shifts in α_t and increases in ψ_{kt} is, of course, somewhat artificial. In reality, most technical change involves some combination of these two effects for some range of tasks. However, it is useful for our theoretical understanding of the effects of technical change to separate these two aspects of technical change. Note that the automation threshold α_t is a technological primitive under Assumption 2, in contrast to the smooth-automation case, where it is an equilibrium object.

Figure 2 plots $\frac{\psi_{kt}}{\psi_{\ell t}}$ when Assumption 2 holds. As we noted above, capital can be used in tasks $\nu \in [0, \alpha_t]$ while labor can be used in all tasks. Whether it is profitable to perform tasks $\nu \in [0, \alpha_t]$ with capital—i.e., automate them—depends on factor prices: automation is profitable when $\frac{R_t}{\psi_{kt}} \leq \frac{W_t}{\psi_{\ell t}}$.

Will this condition hold in equilibrium? Actually, it must. If this condition did not hold, it would not be profitable to use capital in any task. This would mean that capital would sit idle, which is not an equilibrium. The owners of capital would lower the price they charge for capital rather than letting their capital sit idle and yield them no return. We therefore have the following lemma:

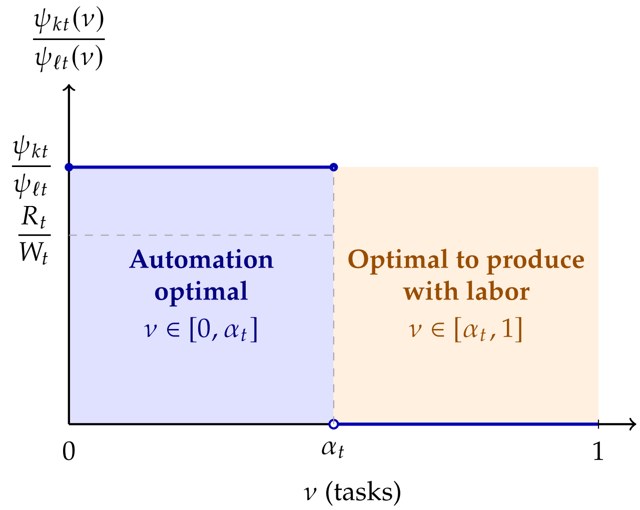


Figure 2. Uniform Productivities.

Note: The figure imposes Assumptions 1 and 2: capital productivity is uniform at ψ_{kt} on $[0, \alpha_t]$ and zero on $[\alpha_t, 1]$ while labor productivity is uniform at $\psi_{\ell t}$ across all tasks. Tasks $\nu \in [\alpha_t, 1]$ can therefore only be produced with labor. Producing tasks $\nu \in [0, \alpha_t]$ with capital is optimal because $\frac{\psi_{kt}}{\psi_{\ell t}}$ exceeds the relative factor price R_t/W_t—the dashed horizontal line.

Lemma 1. *(Factor Prices) In equilibrium, factor prices satisfy*

$$\frac{R_t}{\psi_{kt}} \le \frac{W_t}{\psi_{\ell t}},$$

implying that it is (weakly) profitable to automate tasks $\nu \in [0, \alpha_t]$.

If there is very little capital in the economy relative to the amount of labor in the economy, the relative price of capital will rise to the point where $\frac{R_t}{\psi_{kt}} = \frac{W_t}{\psi_{\ell t}}$. This condition will hold as long as there is not enough capital in the economy to produce all the $Q_t(\nu)$ demanded for tasks $\nu \in [0, \alpha_t]$ when the prices of all tasks are equal (as they are with $\frac{R_t}{\psi_{kt}} = \frac{W_t}{\psi_{\ell t}}$). In this case, labor will be employed in the production of tasks in the range $\nu \in [0, \alpha_t]$ alongside capital. However, if the amount of capital in the economy is large enough relative to labor to produce more than all the quantity of tasks $\nu \in [0, \alpha_t]$ that is demanded when the price of all tasks are equal, then the price of capital will fall to a point where $\frac{R_t}{\psi_{kt}} < \frac{W_t}{\psi_{\ell t}}$. We state this result formally in the following lemma:

Lemma 2. *(Factor Prices) If $\frac{K_t}{L_t} > \underline{K_t/L_t}$, factor prices satisfy*

$$\frac{R_t}{\psi_{kt}} < \frac{W_t}{\psi_{\ell t}},$$

implying that all tasks $v \in [0, \alpha_t]$ are fully automated.

Using the factor market clearing conditions derived below—equation (7)—and $\frac{R_t}{\psi_{kt}} = \frac{W_t}{\psi_{\ell t}}$ we can show that

$$\underline{K_t/L_t} \equiv \frac{\alpha_t}{1-\alpha_t}\frac{\psi_{\ell t}}{\psi_{kt}}.$$

In what follows, we focus on the case where the inequality in Lemma 2 holds, unless otherwise noted.

2.3 Factor Prices

Using the task production function—equation (2)—we have that task-specific demand for labor for tasks $v \in [\alpha_t, 1]$ is $L_t(v) = \frac{Q_t(v)}{\psi_{\ell t}}$. Using the demand for task v—equation (3)—and the task price—equation (5)—we then have that

$$L_t(v) = \begin{cases} \psi_{\ell t}^{\sigma-1} W_t^{-\sigma} Q_t & \text{if } v \in [\alpha_t, 1] \\ 0 & \text{otherwise} \end{cases}.$$

Analogously, for capital, we have that

$$K_t(v) = \begin{cases} \psi_{kt}^{\sigma-1} R_t^{-\sigma} Q_t & \text{if } v \in [0, \alpha_t] \\ 0 & \text{otherwise} \end{cases}.$$

Using the previous two equations and the market clearing conditions from section 1 we have that

$$K_t = \int_0^{\alpha_t} K_t(v)\, dv = \alpha_t \psi_{kt}^{\sigma-1} R_t^{-\sigma} Q_t \quad \text{and} \quad L_t = \int_{\alpha_t}^1 L_t(v)\, dv = (1-\alpha_t)\, \psi_{\ell t}^{\sigma-1} W_t^{-\sigma} Q_t. \tag{6}$$

Solving for the factor prices then yields

$$R_t = \alpha_t^{\frac{1}{\sigma}} \psi_{kt}^{\frac{\sigma-1}{\sigma}} \left(\frac{Q_t}{K_t}\right)^{\frac{1}{\sigma}} \quad \text{and} \quad W_t = (1-\alpha_t)^{\frac{1}{\sigma}}\, \psi_{\ell t}^{\frac{\sigma-1}{\sigma}} \left(\frac{Q_t}{L_t}\right)^{\frac{1}{\sigma}}. \tag{7}$$

To better see how factor supplies shape factor prices, note that we can write the relative factor price as

$$\frac{W_t}{R_t} = \left(\frac{1-\alpha_t}{\alpha_t}\right)^{\frac{1}{\sigma}} \left(\frac{\psi_{kt}}{\psi_{\ell t}}\right)^{\frac{1-\sigma}{\sigma}} \left(\frac{K_t}{L_t}\right)^{\frac{1}{\sigma}}.$$

The relative wage is increasing in the aggregate capital-labor ratio with an elasticity of $\frac{1}{\sigma}$. If tasks are gross complements ($\sigma < 1$), a one-percent increase in the capital-labor ratio raises the relative wage by more than one percent (since $\frac{1}{\sigma} > 1$). Intuitively, capital deepening makes labor relatively more scarce, and, when tasks are gross complements, the market places a disproportionate premium on the scarce factor.

2.4 An Aggregate Production Function

Interestingly, the task-based production function can be represented as a standard CES production function in capital and labor:

Proposition 1. *(Reduced-form Production Function) In equilibrium, output can be represented as a CES-like production function*

$$Q_t = \left[(A_{kt} K_t)^{\frac{\sigma-1}{\sigma}} + (A_{\ell t} L_t)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \tag{8}$$

where

$$A_{kt} = \alpha_t^{\frac{1}{\sigma-1}} \psi_{kt} \quad \text{and} \quad A_{\ell t} = (1-\alpha_t)^{\frac{1}{\sigma-1}} \psi_{\ell t}. \tag{9}$$

Factor shares are given by

$$s_{kt} = \frac{R_t K_t}{Q_t} = \left(\frac{A_{kt} K_t}{Q_t} \right)^{\frac{\sigma-1}{\sigma}} \quad \text{and} \quad s_{\ell t} = \frac{W_t L_t}{Q_t} = \left(\frac{A_{\ell t} L_t}{Q_t} \right)^{\frac{\sigma-1}{\sigma}}. \tag{10}$$

See Appendix B.1 for the proof.

Notice that the labor augmenting and capital augmenting productivity terms A_{kt} and $A_{\ell t}$ are functions of both the automation threshold α_t and of the productivity of capital and labor at the tasks they perform ψ_{kt} and $\psi_{\ell t}$.

As Jones and Tonetti (2026) emphasize, the task-based framework is particularly helpful in interpreting A_{kt} and $A_{\ell t}$. These have traditionally been referred to as capital augmenting and labor augmenting technical change. Whether some particular technological innovation is capital augmenting or labor augmenting is often not an easy question to answer in the abstract. Consider a new type of computer that is more powerful than older models. Does the new computer raise A_{kt}—i.e., is it like having more old computers—or does it perhaps raise $A_{\ell t}$ by economizing on the time of the worker using it. This is not so obvious.

The task-based framework clarifies this. The new computer raises $\psi_{kt}(\nu)$ for the tasks that have been automated to the computer (and it may lead additional tasks to become automated by moving the α_t threshold). Since the tasks performed by the computer and the tasks performed by labor are complements, the new computer will end up raising labor productivity and wages (unless the automation threshold moves too much).

3 Weak Links

Technical change can lead to breathtaking increases in the productivity of particular tasks. In the 18th and early 19th centuries, the cost of spinning cotton fell by two orders of magnitude. In the 20th century, the cost of computing fell by more than six orders of magnitude. Today, AI is reducing the cost of certain tasks to previously unimaginable levels. How do these enormous increases in our ability to perform certain tasks translate into increases in overall output in the economy?

It turns out that the answer to this question depends critically on the degree to which different tasks are substitutable. To illustrate this, we re-write aggregate output—equation (8)—as

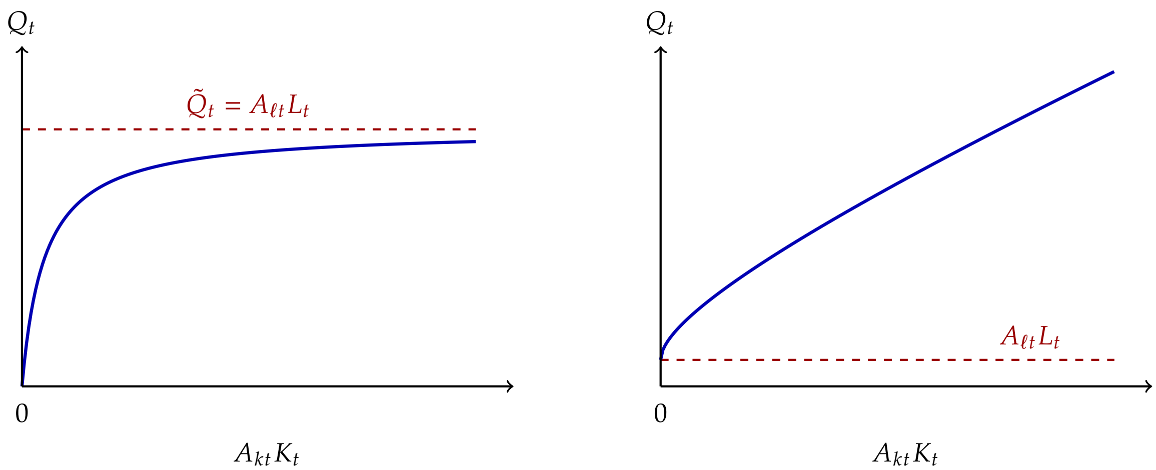
$$Q_t^{\frac{\sigma-1}{\sigma}} = (A_{kt}K_t)^{\frac{\sigma-1}{\sigma}} + (A_{\ell t}L_t)^{\frac{\sigma-1}{\sigma}}. \tag{11}$$

Again, suppose tasks are gross complements ($\sigma < 1$). In this case, if $A_{kt}K_t \to \infty$, then $(A_{kt}K_t)^{\frac{\sigma-1}{\sigma}} \to 0$. Let $\tilde{Q}_t = \lim_{A_{kt}K_t\to\infty} Q_t$, then we have

$$\tilde{Q}_t = A_{\ell t}L_t. \tag{12}$$

This illustrates that, when tasks are gross-complements, aggregate output is limited by the economy's "weakest link." As capital becomes infinitely productive or infinitely abundant, aggregate output remains finite. The left panel in Figure 3 plots aggregate output as $A_{kt}K_t \to \infty$ with $A_{\ell t}L_t = 1$ and $\sigma = 0.5$. As Aghion, Jones and Jones (2019) put it, growth may be constrained "not by what we are good at but rather by what is essential and yet hard to improve."

This result is an instance of Baumol (1967)'s cost disease. In the gross complements case, as we saw above, the price of capital and therefore also the price of the tasks produced by capital falls more than proportionately as the abundance of capital increases. This leads the



(a) Gross Complements ($\sigma < 1$) **(b)** Gross Substitutes ($\sigma > 1$)

Figure 3. Aggregate Output as Capital Becomes Infinitely Productive.

Note: Both panels plot aggregate output $Q_t = \left[(A_{kt}K_t)^{\frac{\sigma-1}{\sigma}} + (A_{\ell t}L_t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$ as a function of $A_{kt}K_t$ holding $A_{\ell t}L_t = 1$. The left panel sets $\sigma = 0.5$: as $A_{kt}K_t \to \infty$, output asymptotes to the weak-links bound $\tilde{Q}_t = A_{\ell t}L_t$ — the dashed line. The right panel sets $\sigma = 2$: output grows without bound, and the level of $A_{\ell t}L_t$ — the dashed line — places no ceiling on output.

economy to become dominated by the tasks performed by labor. The opposite is true if tasks are gross substitutes ($\sigma > 1$). In that case, output explodes, and the labor share of income falls to zero. The right panel of Figure 3 illustrates the case when tasks are gross substitutes.

Next, note that—using equation (10)—we can write

$$A_{kt}K_t = s_{kt}^{\frac{\sigma}{\sigma-1}} Q_t, \tag{13}$$

Using this equation and equations (11) and (12), and (13) we have the following

$$Q_t^{\frac{\sigma-1}{\sigma}} = s_{kt}\,(Q_t)^{\frac{\sigma-1}{\sigma}} + (\tilde{Q}_t)^{\frac{\sigma-1}{\sigma}} \Longrightarrow Q_t = \left(\frac{1}{1-s_{kt}}\right)^{\frac{\sigma}{\sigma-1}} \tilde{Q}_t,$$

Hence, we can show that

$$\frac{\tilde{Q}_t}{Q_t} = \left(\frac{1}{1-s_{kt}}\right)^{\frac{\sigma}{1-\sigma}} \approx 1 + \frac{\sigma}{1-\sigma} s_{kt},$$

where the approximation is valid when the capital share is small, i.e., when $A_{kt}K_t$ is large but finite. This shows that the percent gain in output from making capital infinitely productive at the automated tasks is finite whenever $\sigma < 1$. This gain is the difference between the solid blue line and the broken red line in the left panel of Figure 3.

The model behaves very differently when $\sigma = 1$. In that case, no factor is essential and the gain from $A_{kt}K_t \to \infty$ becomes infinite. When $\sigma > 1$, making a positive measure of tasks infinitely productive is enough to send output to infinity.

4 Output and Productivity

In our model, technical change can take three forms: 1) a change in $\psi_{\ell t}$, 2) a change in ψ_{kt}, and 3) a change in α_t. We now consider how these three forms of technical change affect aggregate output and aggregate productivity in the economy.

Log-differentiating aggregate output—equation (8)—yields

$$\frac{d\ln Q_t}{dt} = s_{kt}\left(\frac{d\ln A_{kt}}{dt} + \frac{d\ln K_t}{dt}\right) + s_{\ell t}\left(\frac{d\ln A_{\ell t}}{dt} + \frac{d\ln L_t}{dt}\right). \tag{14}$$

Rearranging, we have that the Solow (1957) residual is given by

$$\frac{d\ln Q_t}{dt} - s_{kt}\frac{d\ln K_t}{dt} - s_{\ell t}\frac{d\ln L_t}{dt} = s_{kt}\frac{d\ln A_{kt}}{dt} + s_{\ell t}\frac{d\ln A_{\ell t}}{dt}.$$

Using the expressions for A_{kt} and $A_{\ell t}$ in equation (9), we can show that

$$\frac{d\ln A_{kt}}{dt} = \frac{1}{\sigma-1}\frac{d\ln \alpha_t}{dt} + \frac{d\ln \psi_{kt}}{dt} \quad \text{and} \quad \frac{d\ln A_{\ell t}}{dt} = -\frac{1}{\sigma-1}\frac{\alpha_t}{1-\alpha_t}\frac{d\ln \alpha_t}{dt} + \frac{d\ln \psi_{\ell t}}{dt}. \tag{15}$$

To streamline the exposition, define the following

$$\frac{d\ln TFP_t}{dt} = \frac{d\ln Q_t}{dt} - s_{kt}\frac{d\ln K_t}{dt} - s_{\ell t}\frac{d\ln L_t}{dt}.$$

Then, we have

$$\frac{d\ln TFP_t}{dt} = s_{kt}\frac{d\ln \psi_{kt}}{dt} + s_{\ell t}\frac{d\ln \psi_{\ell t}}{dt} + \frac{1}{\sigma-1}\left(s_{kt} - \frac{\alpha_t}{1-\alpha_t}s_{\ell t}\right)\frac{d\ln \alpha_t}{dt}. \tag{16}$$

Using $L_t = (1-\alpha_t)\,\psi_{\ell t}^{\sigma-1} W_t^{-\sigma} Q_t$ and $K_t = \alpha_t \psi_{kt}^{\sigma-1} R_t^{-\sigma} Q_t$, we can express the factor shares as

$$s_{kt} = \alpha_t \left(\frac{R_t}{\psi_{kt}}\right)^{1-\sigma} \quad \text{and} \quad s_{\ell t} = (1-\alpha_t)\left(\frac{W_t}{\psi_{\ell t}}\right)^{1-\sigma}.$$

Hence, we have that

$$s_{kt} - \frac{\alpha_t}{1-\alpha_t} s_{\ell t} = \alpha_t \left[\left(\frac{R_t}{\psi_{kt}}\right)^{1-\sigma} - \left(\frac{W_t}{\psi_{\ell t}}\right)^{1-\sigma}\right].$$

Plugging this into equation (16) we get the following result.

Proposition 2. *(Automation Growth Accounting) The standard Solow (1957) residual is given by*

$$\frac{d\ln TFP_t}{dt} = \underbrace{s_{\ell t}\frac{d\ln \psi_{\ell t}}{dt}}_{\substack{\text{Growth in Productivity}\\ \text{of Labor}}} + \underbrace{s_{kt}\frac{d\ln \psi_{kt}}{dt}}_{\substack{\text{Growth in Productivity}\\ \text{of Capital}}} + \underbrace{\frac{1}{\sigma-1}\left[\left(\frac{R_t}{\psi_{kt}}\right)^{1-\sigma} - \left(\frac{W_t}{\psi_{\ell t}}\right)^{1-\sigma}\right]\frac{d\alpha_t}{dt}}_{\text{Automation}}. \tag{17}$$

This proposition decomposes growth in TFP into the contributions of 1) $\psi_{\ell t}$, i.e., growth in the productivity of labor at tasks performed by labor, 2) ψ_{kt}, i.e., growth in productivity of capital at tasks performed by capital, and 3) α_t, i.e., automation (tasks shifting from labor to capital). In the smooth automation case (Figure 1 and Appendix A), $\frac{R_t}{\psi_{kt}} = \frac{W_t}{\psi_{\ell t}}$ which implies that the automation term is zero. Intuitively, since labor and capital are equally productive when it comes to the marginal task in this case, switching tasks from labor to capital does not affect productivity at the margin. In our simplified model with Lemma 2, however, $\frac{R_t}{\psi_{kt}} < \frac{W_t}{\psi_{\ell t}}$. So, the automation term is positive, i.e., automation increases productivity.

Jones and Tonetti (2026) argue that $\frac{d\ln\psi_{\ell t}}{dt}$ is small and that the vast majority of TFP growth is due to $\frac{d\ln\psi_{kt}}{dt}$. They also emphasize that the automation term in equation (17) being zero in the smooth automation case does not imply that automation is unimportant for TFP growth. After tasks have been automated, they begin benefiting from the high growth in ψ_{kt}. If new tasks were not automated, growth would eventually cease if $\sigma < 1$ for the reasons discussed in section 3: the weight of those tasks in aggregate output would eventually shrivel up to zero. This implies that freezing the set of automated tasks at (say) its 1950 level would have eliminated essentially all of measured TFP growth in the US private business sector since that date.

Holding factor supplies and the task-specific productivities fixed, equations (14) and (15) imply that

$$\frac{d\ln Q_t}{d\ln \alpha_t} = \frac{1}{\sigma-1}\left(s_{kt} - \frac{\alpha_t}{1-\alpha_t}s_{\ell t}\right) = \frac{\alpha_t}{\sigma-1}\left[\left(\frac{R_t}{\psi_{kt}}\right)^{1-\sigma} - \left(\frac{W_t}{\psi_{\ell t}}\right)^{1-\sigma}\right]. \tag{18}$$

As noted above, this expression is positive if and only if $\frac{R_t}{\psi_{kt}} < \frac{W_t}{\psi_{\ell t}}$ — i.e., when Lemma 2 holds. Also, note that this expression is non-negative due to Lemma 1.

5 Factor Prices

We next consider how different forms of technical change as well as changes in factor endowments affect factor prices. We begin by considering the effect of automation. Using the expression we derived above for factor prices—equation (7)—we can derive the following

$$\frac{d\ln R_t}{d\ln \alpha_t} = \frac{1}{\sigma}\left(1 + \frac{d\ln Q_t}{d\ln \alpha_t}\right) \quad \text{and} \quad \frac{d\ln W_t}{d\ln \alpha_t} = \frac{1}{\sigma}\left(\frac{d\ln Q_t}{d\ln \alpha_t} - \frac{\alpha_t}{1-\alpha_t}\right).$$

Since $\frac{d\ln Q_t}{d\ln \alpha_t} \geq 0$, we have that $\frac{d\ln R_t}{d\ln \alpha_t} > 0$. In other words, automation raises the rental rate of capital in the economy. The effect of automation on wages is, however, ambiguous. There are two effects that push in opposite directions. The $-\frac{\alpha_t}{1-\alpha_t}$ term is what Acemoglu and Restrepo (2018) refer to as the displacement effect. It pushes wages down. Intuitively, capital is taking over certain tasks, this lowers labor demand and therefore wages. However, the $\frac{d\ln Q_t}{d\ln \alpha_t}$ term is (weakly) positive, pushing wages up. Acemoglu and Restrepo (2018) refer to this term as the productivity effect. If the displacement effect is larger than the productivity effect, automation will lower wages.

We can combine the expressions for $\frac{d\ln R_t}{d\ln \alpha_t}$ and $\frac{d\ln W_t}{d\ln \alpha_t}$ to get

$$\frac{d\ln(W_t/R_t)}{d\ln \alpha_t} = -\frac{1}{\sigma(1-\alpha_t)} < 0.$$

This shows that automation unambiguously reduces the relative price of labor. The productivity effect is common to both factor prices. It therefore drops out when one looks at the difference between wages and rental rates. The displacement effect, however, hits the two factors asymmetrically. It increases the rental rate (the first term in the formula for

$\frac{d\ln R_t}{d\ln \alpha_t}$), but reduces the wage. Hence, automation always cheapens labor relative to capital in our framework.

Consider next the effect of changes in ψ_{kt} on factor prices. Again using equation (7) we have

$$\frac{d\ln R_t}{d\ln \psi_{kt}} = \frac{1}{\sigma}\left[\frac{d\ln Q_t}{d\ln \psi_{kt}} - (1-\sigma)\right] \quad \text{and} \quad \frac{d\ln W_t}{d\ln \psi_{kt}} = \frac{1}{\sigma}\frac{d\ln Q_t}{d\ln \psi_{kt}}.$$

Using the aggregate production function—equation (8)—we have $\frac{d\ln Q_t}{d\ln \psi_{kt}} = s_{kt}$ which then implies

$$\frac{d\ln R_t}{d\ln \psi_{kt}} = \frac{\sigma - s_{\ell t}}{\sigma} \quad \text{and} \quad \frac{d\ln W_t}{d\ln \psi_{kt}} = \frac{s_{kt}}{\sigma}.$$

We, therefore, have that increases in ψ_{kt} have an unambiguously positive effect on wages. Increases in ψ_{kt} yield a pure productivity effect in Acemoglu and Restrepo (2018)'s terminology. The tasks that are performed by labor are complementary with the tasks that are performed by capital. When capital becomes more productive at the tasks it performs, this raises the marginal product of the tasks performed by labor. This is true not only when $\sigma < 1$. It is true for any (finite) value of σ. But this effect becomes stronger the smaller is the value of σ, i.e., the more complementary are different tasks. You can think of the tasks performed by capital as the machines that the workers are using in their jobs. As these machines become more powerful, the marginal product of the workers rises.

Note that the distinction between α_t and ψ_{kt} captures exactly the most difficult part of thinking about the effect of technical change on labor. When machines become more productive, two things occur. First, the machines take over more tasks. This is α_t. This effect reduces labor demand and can therefore reduce wages. But there is also a second effect. The machines become more powerful tools for the remaining workers. This raises wages. Our model cleanly separates these two effects, which (arguably) is crucial in thinking clearly about the effects of technical change on labor. The ability to separate these two effects is a key advantage of the task-based framework.

What about the effect of an increase in ψ_{kt} on the rental rate of capital. It turns out that this is positive unless $\sigma < s_{\ell t}$. An increase in ψ_{kt} has two effects on capital. First, there is the "direct" effect: the increase in ψ_{kt} makes capital more productive. This effect raises the rental rate. But an increase in ψ_{kt} also makes capital "more abundant" relative to labor in the sense that it increases $A_{kt}K_t$. An increase in ψ_{kt}, therefore, reduces the capital share of income when tasks are gross complements (we show this formally in section 6). This second effect is stronger than the first effect when $\sigma < s_{\ell t}$.

What about effects of $\psi_{\ell t}$? These are analogous to the effects of ψ_{kt} with the roles of the two factors flipped:

$$\frac{d\ln R_t}{d\ln \psi_{\ell t}} = \frac{1}{\sigma}\frac{d\ln Q_t}{d\ln \psi_{\ell t}} \quad \text{and} \quad \frac{d\ln W_t}{d\ln \psi_{\ell t}} = \frac{1}{\sigma}\left[\frac{d\ln Q_t}{d\ln \psi_{\ell t}} - (1-\sigma)\right],$$

$$\frac{d\ln R_t}{d\ln \psi_{\ell t}} = \frac{1-s_{kt}}{\sigma} \quad \text{and} \quad \frac{d\ln W_t}{d\ln \psi_{\ell t}} = \frac{\sigma - s_{kt}}{\sigma}.$$

We can also assess the effect of changes in labor and capital endowments on factor prices in our model. Consider first the effect of an increase in the amount of capital in the economy. Derivations similar to those above yield

$$\frac{d\ln R_t}{d\ln K_t} = \frac{1}{\sigma}\left(\frac{d\ln Q_t}{d\ln K_t} - 1\right) \quad \text{and} \quad \frac{d\ln W_t}{d\ln K_t} = \frac{1}{\sigma}\frac{d\ln Q_t}{d\ln K_t},$$

$$\frac{d\ln R_t}{d\ln K_t} = -\frac{s_{\ell t}}{\sigma} \quad \text{and} \quad \frac{d\ln W_t}{d\ln K_t} = \frac{s_{kt}}{\sigma}.$$

An increase in capital increases the wage unambiguously and reduces the rental rate unambiguously. The effect on the rental rate of capital is only the second of the two effects we discussed when ψ_{kt} increases. I.e., it is a pure "abundance" effect. Analogously, the increase in the wage is a pure "scarcity" effect: labor becomes relatively more scarce when capital increases. This raises the marginal product of labor.

The effect of an increase in the amount of labor is analogous to the effect of an increase in the amount of capital with the roles of the factors reversed

$$\frac{d\ln R_t}{d\ln L_t} = \frac{1}{\sigma}\frac{d\ln Q_t}{d\ln L_t} \quad \text{and} \quad \frac{d\ln W_t}{d\ln L_t} = \frac{1}{\sigma}\left(\frac{d\ln Q_t}{d\ln L_t} - 1\right),$$

$$\frac{d\ln R_t}{d\ln L_t} = \frac{1-s_{kt}}{\sigma} \quad \text{and} \quad \frac{d\ln W_t}{d\ln L_t} = -\frac{s_{kt}}{\sigma}.$$

5.1 Relative Factor Prices

We next consider the effect of technical change and changes in factor endowments on relative task prices and relative factor prices. Consider an arbitrary automated task $\nu_{kt} \in [0, \alpha_t]$ and an arbitrary task performed by labor $\nu_{\ell t} \in [\alpha_t, 1]$. Using the expressions we have derived for relative task prices—equation (4)—and K_t and L_t—equation (6)—we can show that

$$\frac{P_t(\nu_{kt})}{P_t(\nu_{\ell t})} = \left[\frac{\psi_{kt}K_t(\nu_{kt})}{\psi_{\ell t}L_t(\nu_{\ell t})}\right]^{-\frac{1}{\sigma}} \Longrightarrow \frac{P_t(\nu_{kt})}{P_t(\nu_{\ell t})} = \left(\frac{\psi_{kt}}{\psi_{\ell t}}\right)^{-\frac{1}{\sigma}} \cdot \left(\frac{\alpha_t}{1-\alpha_t}\right)^{\frac{1}{\sigma}} \cdot \left(\frac{K_t}{L_t}\right)^{-\frac{1}{\sigma}}.$$

This then implies that

$$\frac{d\ln\left[\frac{P_t(\nu_{kt})}{P_t(\nu_{\ell t})}\right]}{d\ln\left(\psi_{kt}/\psi_{\ell t}\right)} = \frac{d\ln\left[\frac{P_t(\nu_{kt})}{P_t(\nu_{\ell t})}\right]}{d\ln\left(\frac{K_t}{L_t}\right)} = -\frac{1}{\sigma}.$$

This shows that when tasks are gross complements ($\sigma < 1$), technical change and changes in factor endowments trigger Baumol (1967) cost-disease logic. Improving the relative quality of capital creates a bottleneck in the labor-intensive tasks, which more than proportionately decreases the relative price of the tasks performed by capital. The same occurs with capital deepening.

Using the expression for the task-specific prices—equation (5) —we have that

$$P_t(\nu_{kt}) = \frac{R_t}{\psi_{kt}} \quad \text{and} \quad P_t(\nu_{\ell t}) = \frac{W_t}{\psi_{\ell t}},$$

which implies that

$$\frac{P_t(\nu_{kt})}{P_t(\nu_{\ell t})} = \frac{R_t}{W_t} \cdot \frac{\psi_{\ell t}}{\psi_{kt}}.$$

Hence, we have the following

$$\frac{d\ln\left(W_t/R_t\right)}{d\ln\left(\psi_{kt}/\psi_{\ell t}\right)} = -\frac{d\ln\left[\frac{P_t(\nu_{kt})}{P_t(\nu_{\ell t})}\right]}{d\ln\left(\psi_{kt}/\psi_{\ell t}\right)} - 1 = \frac{1-\sigma}{\sigma} \quad \text{and} \quad \frac{d\ln\left(W_t/R_t\right)}{d\ln\left(K_t/L_t\right)} = -\frac{d\ln\left[\frac{P_t(\nu_{kt})}{P_t(\nu_{\ell t})}\right]}{d\ln\left(K_t/L_t\right)} = \frac{1}{\sigma}.$$

When tasks are gross complements ($\sigma < 1$), improvements in the relative productivity of capital increases the relative price of labor. This flows from the same bottleneck / weak-links logic. Capital deepening, however, increases the relative price of labor for any value of σ (but by more the lower is σ).

6 Factor Shares

The last variable we consider is the labor share of income. Recall from equation (10) that the labor share is

$$s_{\ell t} = \left(\frac{A_{\ell t} L_t}{Q_t}\right)^{\frac{\sigma-1}{\sigma}}.$$

Taking the total log-derivative of this equation, we get

$$d \ln s_{\ell t} = \frac{\sigma - 1}{\sigma}\left(d \ln A_{\ell t} + d \ln L_t - d \ln Q_t\right).$$

Using the expression for $\frac{d \ln Q_t}{dt}$—equation (14)—and the fact that $s_{\ell t} = 1 - s_{kt}$, we can express $d \ln s_{\ell t}$ as

$$d \ln s_{\ell t} = \frac{\sigma - 1}{\sigma} s_{kt}\left[d \ln A_{\ell t} - d \ln A_{kt} - d \ln\left(K_t/L_t\right)\right],$$

Next, using the expressions for $\frac{d \ln A_{kt}}{dt}$ and $\frac{d \ln A_{\ell t}}{dt}$—equation (15)—we get that

$$d \ln s_{\ell t} = s_{kt}\left(\frac{1-\sigma}{\sigma}\right)\left[d \ln \psi_{kt} - d \ln \psi_{\ell t} + d \ln\left(K_t/L_t\right)\right] - \left(\frac{s_{kt}}{\sigma}\right)\left(\frac{1}{1-\alpha_t}\right) d \ln \alpha_t.$$

Hence, we can summarize the impact of exogenous shocks on the labor share as follows

$$\frac{d \ln s_{\ell t}}{d \ln \psi_{kt}} = -\frac{d \ln s_{\ell t}}{d \ln \psi_{\ell t}} = \frac{d \ln s_{\ell t}}{d \ln\left(K_t/L_t\right)} = s_{kt}\left(\frac{1-\sigma}{\sigma}\right) \quad \text{and} \quad \frac{d \ln s_{\ell t}}{d \ln \alpha_t} = -\left(\frac{s_{kt}}{\sigma}\right)\left(\frac{1}{1-\alpha_t}\right) < 0.$$

Automation unambiguously reduces the labor share. As emphasized by Aghion, Jones and Jones (2019) and Jones and Tonetti (2026), automation is simultaneously labor augmenting and capital depleting: An increase in α_t spreads the fixed capital stock across more tasks — lowering A_{kt} — while concentrating the fixed labor supply on fewer tasks — raising $A_{\ell t}$.

When tasks are gross complements ($\sigma < 1$), $s_{kt}\left(\frac{1-\sigma}{\sigma}\right) > 0$. In this case then, an increase in ψ_{kt} raises the labor share. This is what we saw happen in section 3 as we took $A_{kt}K_t \to \infty$. In the gross complements case, as capital becomes more abundant, its relative price falls more than proportionately, which leads the capital share to fall and the labor share to rise.

7 Stability of Capital Share

Uzawa (1961)'s theorem says that, on a balanced growth path, technological progress must be purely labor augmenting. In our framework, this means that A_{kt} must be constant in the long run on a balanced growth path, whereas $A_{\ell t}$ will grow at a constant rate. Prior to the emergence of the task-based framework, this theorem seemed puzzling. As Acemoglu (2009, p. 59) put it: "This result is very surprising and troubling, since there are no compelling reasons for why technological progress should take this form." Arguably, the surprising and troubling nature of the Uzawa (1961) theorem at that time was a reflection of how poor our understanding was of what it meant for technological process to be purely labor augmenting prior to the emergence of the task-based framework.

Jones and Liu (2024) use a task-based framework to clarify the nature of Uzawa's theorem. They show that all technical progress can be embodied in capital and Uzawa's theorem can nonetheless hold. Recall that, in our model above, $A_{kt} = \alpha_t^{\frac{1}{\sigma-1}} \psi_{kt}$. Thus, increases in α_t raise the capital share (unambiguously), while increases in ψ_{kt} lower the capital share when $\sigma < 1$. Both of these forms of technical change are improvements in the productivity of capital. If they occur in the right proportion, A_{kt} will remain constant (as long as $\sigma < 1$) .

To flesh this out, we start by relaxing the assumption that capital is produced ex nihilo, and assume instead that capital is an intermediate input that is produced using the final good. The only thing that changes in the environment is the final good's resource constraint, which becomes

$$Q_t = C_t + \int_0^1 K_t(\nu)\, d\nu.$$

An immediate implication is that $R_t = 1$ since we take the final good as the numeraire. The task-specific capital demand is then given by

$$K_t(\nu) = \begin{cases} \psi_{kt}^{\sigma-1} Q_t & \text{if } \nu \in [0, \alpha_t] \\ 0 & \text{otherwise} \end{cases}.$$

It is useful to define the harmonic average of the capital-embodied technology terms as

$$Z_{kt} = \left(\frac{1}{\alpha_t} \int_0^{\alpha_t} \psi_{kt}(\nu)^{\sigma-1}\, d\nu \right)^{-1}.$$

Note that in our case where $\psi_{kt}(\nu) = \psi_{kt}$ for all $\nu \in [0, \alpha_t]$, we have $Z_{kt} = \psi_{kt}^{1-\sigma}$. We can also express Z_{kt} in terms of the reduced-form capital-augmenting technology as

$$Z_{kt} = \alpha_t A_{kt}^{1-\sigma},$$

where we used $A_{kt} = \alpha_t^{\frac{1}{\sigma-1}} \psi_{kt}$. Using the resource constraint for capital, we can show that

$$K_t = \alpha_t \underbrace{\psi_{kt}^{\sigma-1}}_{=(Z_{kt})^{-1}} Q_t \Longrightarrow K_t = \alpha_t \left(Z_{kt}\right)^{-1} Q_t.$$

The capital share is then given by

$$s_{kt} = \frac{K_t}{Q_t} = \alpha_t \left(Z_{kt}\right)^{-1} = \alpha_t \psi_{kt}^{\sigma-1}$$

This illustrates Jones and Liu (2024)'s central result. The capital share is increasing in the fraction of automated tasks and declining in the harmonic average of the capital-embodied technology terms, which in our simple model means that it is declining in ψ_{kt} as long as $\sigma < 1$. Hence, a stable capital share is possible when these two effects exactly offset each other.

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Appendix

A Smooth Automation

Throughout this section, we replace Assumptions 1 and 2 with the following assumption.

Assumption 3. *(Smooth Automation) We assume that*

$$\frac{\psi_{kt}(\nu)}{\psi_{\ell t}(\nu)}$$

is decreasing in ν so that labor has a comparative advantage at high-index tasks. Then, we can define the following cutoff

$$\frac{W_t}{\psi_{\ell t}(\alpha_t)} = \frac{R_t}{\psi_{kt}(\alpha_t)}$$

so that tasks $\nu \in [0, \alpha_t]$ are automated and performed using capital.

Proposition 3. *(Smooth-Automation Growth Accounting) When automation is smooth and under the assumption of constant returns to scale, TFP growth is given by*

$$\frac{d\ln TFP_t}{dt} = \underbrace{s_{\ell t}\int_{\alpha_t}^{1}\left(\frac{\psi_{\ell t}(\nu)}{A_{\ell t}}\right)^{\sigma-1}\frac{d\ln\psi_{\ell t}(\nu)}{dt}d\nu}_{\substack{\text{Growth in Productivity}\\ \text{of Labor}}} + \underbrace{s_{kt}\int_{0}^{\alpha_t}\left(\frac{\psi_{kt}(\nu)}{A_{kt}}\right)^{\sigma-1}\frac{d\ln\psi_{kt}(\nu)}{dt}d\nu}_{\substack{\text{Growth in Productivity}\\ \text{of Capital}}}.$$

Proposition 3 shows that the effect of automation on growth works through the channel of improving machines. This boost only applies to a fraction of tasks.

B Proofs and Derivations

B.1 Proof of Proposition 1

Proof. The ideal-price index can be unpacked as follows

$$1 = \int_0^{\alpha_t}\left(\frac{R_t}{\psi_{kt}}\right)^{1-\sigma}d\nu + \int_{\alpha_t}^{1}\left(\frac{W_t}{\psi_{\ell t}}\right)^{1-\sigma}d\nu,$$

which can be re-written as

$$1 = R_t^{1-\sigma}\psi_{kt}^{\sigma-1}\alpha_t + W_t^{1-\sigma}\left(1-\alpha_t\right)\psi_{\ell t}^{\sigma-1}.$$

Next, combining the labor and capital resource constraint with their corresponding task-specific demand yields

$$R_t = \alpha_t^{\frac{1}{\sigma}}\psi_{kt}^{\frac{\sigma-1}{\sigma}}\left(\frac{Q_t}{K_t}\right)^{\frac{1}{\sigma}} \quad \text{and} \quad W_t = \left(1-\alpha_t\right)^{\frac{1}{\sigma}}\psi_{\ell t}^{\frac{\sigma-1}{\sigma}}\left(\frac{Q_t}{L_t}\right)^{\frac{1}{\sigma}}.$$

Therefore, we have

$$1 = \alpha_t^{\frac{1-\sigma}{\sigma}}\psi_{kt}^{\frac{(\sigma-1)(1-\sigma)}{\sigma}}\left(\frac{Q_t}{K_t}\right)^{\frac{1-\sigma}{\sigma}}\psi_{kt}^{\sigma-1}\alpha_t + \left(1-\alpha_t\right)^{\frac{1-\sigma}{\sigma}}\psi_{\ell t}^{\frac{(\sigma-1)(1-\sigma)}{\sigma}}\left(\frac{Q_t}{L_t}\right)^{\frac{1-\sigma}{\sigma}}\left(1-\alpha_t\right)\psi_{\ell t}^{\sigma-1},$$

which can be simplified to

$$Q_t = \left[\alpha_t^{\frac{1}{\sigma}}\psi_{kt}^{\frac{\sigma-1}{\sigma}}K_t^{\frac{\sigma-1}{\sigma}} + \left(1-\alpha_t\right)^{\frac{1}{\sigma}}\psi_{\ell t}^{\frac{\sigma-1}{\sigma}}L_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

Let's define

$$A_{kt} = \alpha_t^{\frac{1}{\sigma-1}}\psi_{kt} \quad \text{and} \quad A_{\ell t} = \left(1-\alpha_t\right)^{\frac{1}{\sigma-1}}\psi_{\ell t}.$$

Then, we can write

$$Q_t = \left[\left(A_{kt}K_t\right)^{\frac{\sigma-1}{\sigma}} + \left(A_{\ell t}L_t\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

Next, recall that

$$R_t = \alpha_t^{\frac{1}{\sigma}}\psi_{kt}^{\frac{\sigma-1}{\sigma}}\left(\frac{Q_t}{K_t}\right)^{\frac{1}{\sigma}} \quad \text{and} \quad W_t = \left(1-\alpha_t\right)^{\frac{1}{\sigma}}\psi_{\ell t}^{\frac{\sigma-1}{\sigma}}\left(\frac{Q_t}{L_t}\right)^{\frac{1}{\sigma}}.$$

We can re-write these as

$$\frac{R_tK_t}{Q_t} = \alpha_t^{\frac{1}{\sigma}}\psi_{kt}^{\frac{\sigma-1}{\sigma}}\left(\frac{K_t}{Q_t}\right)^{\frac{\sigma-1}{\sigma}} \quad \text{and} \quad \frac{W_tL_t}{Q_t} = \left(1-\alpha_t\right)^{\frac{1}{\sigma}}\psi_{\ell t}^{\frac{\sigma-1}{\sigma}}\left(\frac{L_t}{Q_t}\right)^{\frac{\sigma-1}{\sigma}}.$$

Finally, using the definitions of A_{kt} and $A_{\ell t}$, we get the factor shares

$$s_{kt} = \frac{R_t K_t}{Q_t} = \left(\frac{A_{kt} K_t}{Q_t}\right)^{\frac{\sigma-1}{\sigma}} \quad \text{and} \quad s_{\ell t} = \frac{W_t L_t}{Q_t} = \left(\frac{A_{\ell t} L_t}{Q_t}\right)^{\frac{\sigma-1}{\sigma}}.$$

■

B.2 Proof of Proposition 3

Proof. Recall that the ideal-price index is given by

$$1 = \left[\int_0^1 P_t(\nu)^{1-\sigma}\, d\nu\right]^{\frac{1}{1-\sigma}}.$$

This can be unpacked as follows

$$1 = \int_0^{\alpha_t} \left(\frac{R_t}{\psi_{kt}(\nu)}\right)^{1-\sigma} d\nu + \int_{\alpha_t}^1 \left(\frac{W_t}{\psi_{\ell t}(\nu)}\right)^{1-\sigma} d\nu,$$

which can be re-written as

$$1 = R_t^{1-\sigma}\left(\int_0^{\alpha_t} \psi_{kt}(\nu)^{\sigma-1}\, d\nu\right) + W_t^{1-\sigma}\left(\int_{\alpha_t}^1 \psi_{\ell t}(\nu)^{\sigma-1}\, d\nu\right).$$

Next, combining the labor and capital resource constraint with their corresponding task-specific demand yields

$$R_t = \left(\int_0^{\alpha_t} \psi_{kt}(\nu)^{\sigma-1}\, d\nu\right)^{\frac{1}{\sigma}} \left(\frac{Q_t}{K_t}\right)^{\frac{1}{\sigma}} \quad \text{and} \quad W_t = \left(\int_{\alpha_t}^1 \psi_{\ell t}(\nu)^{\sigma-1}\, d\nu\right)^{\frac{1}{\sigma}} \left(\frac{Q_t}{L_t}\right)^{\frac{1}{\sigma}}.$$

Therefore, we have

$$1 = \left(\int_0^{\alpha_t} \psi_{kt}(\nu)^{\sigma-1}\, d\nu\right)^{\frac{1}{\sigma}} \left(\frac{Q_t}{K_t}\right)^{\frac{1-\sigma}{\sigma}} + \left(\int_{\alpha_t}^1 \psi_{\ell t}(\nu)^{\sigma-1}\, d\nu\right)^{\frac{1}{\sigma}} \left(\frac{Q_t}{L_t}\right)^{\frac{1-\sigma}{\sigma}},$$

which can be simplified to

$$Q_t = \left[\left(\int_0^{\alpha_t} \psi_{kt}(\nu)^{\sigma-1} d\nu \right)^{\frac{1}{\sigma}} K_t^{\frac{\sigma-1}{\sigma}} + \left(\int_{\alpha_t}^1 \psi_{\ell t}(\nu)^{\sigma-1} d\nu \right)^{\frac{1}{\sigma}} L_t^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}.$$

Let's define

$$A_{kt} = \left(\int_0^{\alpha_t} \psi_{kt}(\nu)^{\sigma-1} d\nu \right)^{\frac{1}{\sigma-1}} \quad \text{and} \quad A_{\ell t} = \left(\int_{\alpha_t}^1 \psi_{\ell t}(\nu)^{\sigma-1} d\nu \right)^{\frac{1}{\sigma-1}}.$$

Then, we can write

$$Q_t = \left[(A_{kt} K_t)^{\frac{\sigma-1}{\sigma}} + (A_{\ell t} L_t)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}.$$

Log-differentiating aggregate output yields

$$\frac{d \ln Q_t}{dt} = s_{kt} \left(\frac{d \ln A_{kt}}{dt} + \frac{d \ln K_t}{dt} \right) + s_{\ell t} \left(\frac{d \ln A_{\ell t}}{dt} + \frac{d \ln L_t}{dt} \right),$$

which implies that the Solow (1957) residual is given by

$$\frac{d \ln Q_t}{dt} - s_{kt} \frac{d \ln K_t}{dt} - s_{\ell t} \frac{d \ln L_t}{dt} = s_{kt} \frac{d \ln A_{kt}}{dt} + s_{\ell t} \frac{d \ln A_{\ell t}}{dt}.$$

Using the expressions for A_{kt} and $A_{\ell t}$, we can show that

$$\frac{d \ln A_{kt}}{dt} = (A_{kt})^{1-\sigma} \left[\frac{1}{\sigma - 1} \alpha_t \psi_{kt}(\alpha_t)^{\sigma-1} \frac{d \ln \alpha_t}{dt} + \int_0^{\alpha_t} \psi_{kt}(\nu)^{\sigma-1} \frac{d \ln \psi_{kt}(\nu)}{dt} d\nu \right],$$

and

$$\frac{d \ln A_{\ell t}}{dt} = (A_{\ell t})^{1-\sigma} \left[-\frac{1}{\sigma - 1} \alpha_t \psi_{\ell t}(\alpha_t)^{\sigma-1} \frac{d \ln \alpha_t}{dt} + \int_{\alpha_t}^1 \psi_{\ell t}(\nu)^{\sigma-1} \frac{d \ln \psi_{\ell t}(\nu)}{dt} d\nu \right].$$

Then, we have

$$\frac{d\ln TFP_t}{dt} = \underbrace{s_{\ell t}\int_{\alpha_t}^{1}\left(\frac{\psi_{\ell t}(\nu)}{A_{\ell t}}\right)^{\sigma-1}\frac{d\ln\psi_{\ell t}(\nu)}{dt}d\nu}_{\substack{\text{Growth in Productivity}\\ \text{of Labor}}} + \underbrace{s_{kt}\int_{0}^{\alpha_t}\left(\frac{\psi_{kt}(\nu)}{A_{kt}}\right)^{\sigma-1}\frac{d\ln\psi_{kt}(\nu)}{dt}d\nu}_{\substack{\text{Growth in Productivity}\\ \text{of Capital}}}$$
$$+\underbrace{\frac{\alpha_t}{\sigma-1}\left[s_{kt}\left(\frac{\psi_{kt}(\alpha_t)}{A_{kt}}\right)^{\sigma-1} - s_{\ell t}\left(\frac{\psi_{\ell t}(\alpha_t)}{A_{\ell t}}\right)^{\sigma-1}\right]\frac{d\ln\alpha_t}{dt}}_{\text{Automation}}.$$

And the factor shares are still given by

$$s_{kt} = \left(\frac{A_{kt}K_t}{Q_t}\right)^{\frac{\sigma-1}{\sigma}} \quad \text{and} \quad s_{\ell t} = \left(\frac{A_{\ell t}L_t}{Q_t}\right)^{\frac{\sigma-1}{\sigma}}.$$

Note that from Assumption 3, we have

$$\frac{W_t}{\psi_{\ell t}(\alpha_t)} = \frac{R_t}{\psi_{kt}(\alpha_t)}.$$

This can be re-written as

$$\frac{s_{kt}}{s_{\ell t}} = \frac{\psi_{kt}(\alpha_t)K_t}{\psi_{\ell t}(\alpha_t)L_t}.$$

From the resource constraints and the capital and labor demand, we can show that

$$\frac{K_t}{L_t} = \left(\frac{A_{kt}}{A_{\ell t}}\right)^{\sigma-1}\left(\frac{R_t}{W_t}\right)^{-\sigma}.$$

Recall that we have $\frac{R_t}{W_t} = \frac{\psi_{kt}(\alpha_t)}{\psi_{\ell t}(\alpha_t)}$ by assumption, then we showed that

$$\frac{K_t}{L_t} = \left(\frac{A_{kt}}{A_{\ell t}}\right)^{\sigma-1}\left(\frac{\psi_{kt}(\alpha_t)}{\psi_{\ell t}(\alpha_t)}\right)^{-\sigma} \Longrightarrow \frac{s_{kt}}{s_{\ell t}} = \left(\frac{A_{kt}}{A_{\ell t}}\right)^{\sigma-1}\left(\frac{\psi_{kt}(\alpha_t)}{\psi_{\ell t}(\alpha_t)}\right)^{1-\sigma},$$

which can be written as

$$s_{kt}\left(\frac{\psi_{kt}(\alpha_t)}{A_{kt}}\right)^{\sigma-1} = s_{\ell t}\left(\frac{\psi_{\ell t}(\alpha_t)}{A_{\ell t}}\right)^{\sigma-1},$$

which implies that the automation effect is zero when automation is smooth. ■