The Expanding Variety Model

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Allocation of Resources to Innovation

- Last lecture, we assumed

$$L_A(t) = sL(t)$$

- This was a short cut
- Similar to constant savings rate in Solow model
- Now we will study the allocation of resources to innovation

Excludability of Knowledge

- Last lecture, we emphasized non-rival nature of knowledge
- While knowledge is non-rival, much knowledge is excludable
- Excludability: Ability to prevent someone from using something
- Sources of excludability:
  - Patents (but not all knowledge is patentable)
  - Trade secrets (reverse-engineering can limit secrecy)
  - Difficulty (some things are hard learn)
- The excludability of knowledge implies that knowledge can be produced for profit

Innovation and Imperfect Competition

- Perfect competition unlikely to yield efficent level of innovation
- With perfect competition, the price of an item is equal to its marginal cost
- The marginal cost of using an existing idea is zero
- Rental price of existing knowledge should thus be zero
  - Think of the licensing fee for a drug formula
- But if price of existing knowledge is zero, there is no incentive to create knowledge

Fundamental Innovation Trade-Off

- For efficient use, price of existing knowledge should be zero
  - This creates too little incentive to innovate
- For innovation to occur, price of existing knowledge needs to be positive (i.e., above marginal cost)
  - This yields too little use of existing knowledge (i.e., too few people can afford a drug)
- Laissez faire economic policy doesn't work well for innovation

Roadblock for Economic Theory

- Inadequacy of perfect competition for the economics of innovation was a major roadblock for economic theory
- In 1960s, economists were good at building perfectly competitive models, but not good at building models with imperfect competition
- Major step forward: Monopolistic competition framework of Dixit and Stiglitz (1977)
- Has become a basic building block of:
  - Economic growth models (e.g., Romer 90)
  - International trade models (e.g., Krugman 79)
  - New Keynesian models (e.g., Blanchard-Kiyotaki 87)

The Dixit-Stiglitz Model

- Continuum of firms i of measure N
- Each firm is the monopoly supplier of a differentiated product
- These products enter household utility through the consumption index

$$C = \left[\int_0^N c_i^{\frac{\phi-1}{\phi}} di \right]^{\frac{\phi}{\phi-1}}$$

- Household utility is then $U(C, L, ...)$ where C is the index above
- ϕ is the elasticity of substitution between the different c_i

The Dixit-Stiglitz Model

- Suppose the price of the good i is p_i
- Household would like to maximize the amount of C it can purchase for a given amount of spending Z
- It therefore solves:

$$\max_{c_i} \left[\int_0^N c_i^{\frac{\phi-1}{\phi}} di \right]^{\frac{\phi}{\phi-1}} \quad \text{subject to} \quad \int_0^N p_i c_i di = Z$$

- We can form a Lagrangian:

$$L = \left[\int_0^N c_i^{\frac{\phi-1}{\phi}} di \right]^{\frac{\phi}{\phi-1}} - \lambda \left[\int_0^N p_i c_i di - Z \right]$$

- Differentiating with respect to c_i yields:

$$\left(\frac{C}{c_i}\right)^{\frac{1}{\phi}} = \lambda p_i$$

- This is true for each i. Divide the one for i by the one for i':

$$\left(\frac{c_i'}{c_i}\right)^{\frac{1}{\phi}} = \frac{p_i}{p_i'}$$

- Rearranging yields:

$$c_i = c_i' \left(\frac{p_i}{p_i'}\right)^{-\phi}$$

- Shows that price elasticity of demand is ϕ

The Dixit-Stiglitz Model

- Let's define the ideal price index P as the minimum expenditure needed to purchase 1 unit of the consumption index
- Some additional algebra then yields see steps

$$P = \left[\int_0^N p_i^{1-\phi} di \right]^{\frac{1}{1-\phi}}$$

- Using the fact that $\lambda = 1/P$ yields

$$c_i = C \left(\frac{p_i}{P} \right)^{-\phi}$$

which is just another way to express the demand curve for c_i

The Dixit-Stiglitz Model

- Household preferences display "love of variety"
- Suppose the price of all the goods is equal to p
- Price index is then

$$P = \left[\int_0^N p^{1-\phi} di \right]^{\frac{1}{1-\phi}} = p \left[\int_0^N 1 di \right]^{\frac{1}{1-\phi}} = pN^{-\frac{1}{\phi-1}}$$

- If $\phi > 1$, P is falling in N
- Households get more C per unit spending as N increases

The Dixit-Stiglitz Model

- Let's now return to the firms
- Suppose their marginal cost of production is ψ
- Firm profits are then given by $\Pi_i = p_i c_i - \psi c_i$
- Firms set prices to maximize profits given demand for their product

$$\max_{p_i} C \left(\frac{p_i}{P} \right)^{-\phi} (p_i - \psi)$$

- Profit maximization yields

$$p_i = \frac{\phi}{\phi - 1} \psi$$

- Firm's set prices equal to a markup over marginal cost
- For markup to be finite, ϕ must be larger than 1

Dixit-Stiglitz Model

- Tractable general equilibrium framework where firms have market power and can set prices

- Can also be applied to factor markets

- Production function:

$$Y = \left[\int_0^N y_i^{\frac{\phi-1}{\phi}} di \right]^{\frac{\phi}{\phi-1}}$$

  where y_i are intermediate inputs

- In this case producer of intermediate input is a monopolist with market power

The Expanding Variety Model of Growth

- Let's now consider the expanding variety model of growth
- Original version due to Romer (1990)
- Model has three classes of agents:
  - Households
  - Final-goods producing firms
  - Intermediate-goods producing / R&D firms
- We consider these in turn

HOUSEHOLDS

- Constant population of households that consume and supply labor
- Households supply an aggregate quantity L of labor inelastically
- Households own all firms in equal proportions
- Household utility

$$U = \int_0^\infty \exp(-\rho t) \frac{C(t)^{1-\theta}}{1-\theta} dt$$

- As in Ramsey model, household optimization yields:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r(t) - \rho)$$

Final Goods Producing Firms

- Final goods are produced in a perfectly competitive market with the production function

$$Y(t) = \frac{1}{1-\beta} L_Y(t)^{\beta} \int_0^{N(t)} x(i,t)^{1-\beta} di$$

- Inputs to final goods production:
  - Labor: $L_Y(t)$
  - $N(t)$ distinct intermediate inputs: $x(i,t)$
- Notice that final goods production is constant returns to scale in physical inputs

Final Goods Producing Firms

- Notice that production function can also be written

$$Y(t) = \frac{1}{1-\beta} L_Y(t)^{\beta} \mathbf{X}(t)^{1-\beta}$$

  where

$$\mathbf{X}(t) = \left[\int_0^{N(t)} x(i,t)^{\frac{\phi-1}{\phi}} di \right]^{\frac{\phi}{\phi-1}}$$

  and $\phi = 1/\beta$

- So, intermediate input part of production function takes Dixit-Stiglitz form

Final Goods Producing Firms

$$Y(t) = \frac{1}{1-\beta} L_Y(t)^\beta \int_0^{N(t)} x(i,t)^{1-\beta} di$$

- Production is additively separable in different $x(i,t)$s
- Marginal product of each $x(i,t)$ independent of the others
- New products don't make older products obsolete
  (strong contrast with "quality ladder model")

Final Goods Producing Firms

- Final goods firms maximize profits

$$\Pi = \frac{1}{1-\beta} L_Y(t)^{\beta} \int_0^{N(t)} x(i,t)^{1-\beta} di - \int_0^{N(t)} p(i,t) x(i,t) di - w(t) L_Y(t)$$

  where $p(i,t)$ is the price of intermediate input $x(i,t)$

- Intermediate input demand:

$$L_Y(t)^{\beta} x(i,t)^{-\beta} - p(i,t) = 0$$

  and rearranging:

$$x(i,t) = p(i,t)^{-1/\beta} L_Y(t)$$

- Labor demand:

$$\beta \frac{Y(t)}{L_Y(t)} = w(t)$$

Intermediate Goods Producers / R&D Firms

- This is the real heart of the model!
- There is free entry into development of new intermediate inputs
- Once a firm develops a new intermediate input, it gains a perpetual monopoly over this product (either through a patent or secrecy)
- Firm then sells the product at a markup over marginal cost forever, earning a profit that allows it to recoup development cost

Intermediate Goods Producers

- Let's start by considering the pricing decision and profits of the firm once it has developed the product
- Suppose intermediate i is produced simply using ψ units of final good
- Let's make the final good the numeraire (i.e., price of final good is 1)
- This means marginal cost of producing intermediate i is ψ
- Flow profit:

$$\Pi(i, t) = p(i, t)x(i, t) - \psi x(i, t)$$

Intermediate Goods Producers

- Plugging demand into profits we get

$$\Pi(i,t) = p(i,t)^{-1/\beta} L_Y(t)[p(i,t) - \psi]$$

- Differentiating to find profit maximizing price:

$$\left(-\frac{1}{\beta} + 1\right) p(i,t)^{-\frac{1}{\beta}} + \frac{1}{\beta} p(i,t)^{-\frac{1}{\beta}-1} \psi = 0$$

- Rearranging yields

$$p(i,t) = \frac{1}{1-\beta}\psi$$

Final Good Output

- Let's normalize $\psi = (1 - \beta)$
- Implies that

$$p(i,t) = 1$$

- This means that

$$x(i,t) = p(i,t)^{-1/\beta} L_Y(t) = L_Y(t)$$

- Final good output then becomes

$$\begin{aligned} Y(t) &= \frac{1}{1-\beta} L_Y(t)^{\beta} \int_0^{N(t)} x(i,t)^{1-\beta} di \\ &= \frac{1}{1-\beta} L_Y(t)^{\beta} \int_0^{N(t)} L_Y(t)^{1-\beta} di \\ &= \frac{1}{1-\beta} N(t) L_Y(t) \end{aligned}$$

Final Good Output

$$Y(t) = \frac{1}{1-\beta} N(t) L_Y(t)$$

- $N(t)$ (# of intermediate goods invented) acts like "productivity"
- Product innovation raises aggregate output
- Different flavors of the model:
  - Could be consumer products, rather than intermediate inputs
  - Could be "machines" or processes (process innovation)

R&D DECISION

- In this model, innovation is profit driven
- Since there is free entry, people will innovate to the point where marginal cost is equal to marginal profit
- Flow profit associated with successful innovation:

$$\begin{aligned}\Pi(i,t) &= p(i,t)x(i,t) - \psi x(i,t) \\ &= L_Y(t) - (1-\beta)L_Y(t) \\ &= \beta L_Y(t)\end{aligned}$$

- The total value of owning the right to sell intermediate i is

$$V(i,t) = \int_t^\infty \exp\left(- \int_t^s r(s')ds' \right) \Pi(i,s)ds$$

- If $r(t) = r$ – which turns out to be the case – and using expression for profits on last slide, this simplifies to

$$V(t) = \int_t^\infty \exp(-r(s-t))\beta L_Y(t)ds$$

- This is just the discounted value of the profits

R&D Production Function

- R&D production function:

$$\dot{N}(t) = \eta N(t) L_R(t)$$

- This is the $\phi = 1$ case from last lecture as in Romer (1990)
- Alternative cases:
  - Semi-endogenous growth model:

$$\dot{N}(t) = \eta N(t)^{\phi} L_R(t) \qquad \text{with} \qquad \phi < 1$$

  - "Lab equipment" model

$$\dot{N}(t) = \eta Z(t)$$

    where $Z(t)$ are final goods (this is a $\phi = 1$ model.)

R&D Decision

- Hiring one R&D worker yields $\eta N(t)$ new products
- Marginal benefit of hiring R&D workers: $\eta N(t)V(i,t)$
- Marginal cost of hiring R&D workers: $w(t)$
- Free entry therefore implies

$$\eta N(t)V(i,t) = w(t)$$

Balanced Growth Path

- We look for an equilibrium with a constant growth rate g for consumption and output
- In such an equilibrium, the interest rate must be constant:

$$g = \frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r - \rho)$$

- We conjecture that $L_R(t) = L_R$ and $L_Y(t) = L_Y$
- This implies

$$V = \int_t^{\infty} \exp(-r(s-t))\beta L_Y(t) ds = \frac{\beta L_Y}{r}$$

Balanced Growth Path

- Recall that labor supply is given by

$$\beta \frac{Y(t)}{L_Y(t)} = w(t)$$

- The value of the intermediate firm is

$$V = \frac{\beta L_Y}{r}$$

- Plugging these in for $V(i,t)$ and $w(t)$ in the free entry condition yields

$$\eta N(t) V(i,t) = w(t) \quad => \quad \eta N(t) \frac{\beta L_Y}{r} = \beta \frac{Y(t)}{L_Y}$$

Balanced Growth Path

- Recall that output of final goods is

$$Y(t) = \frac{1}{1-\beta} N(t) L_Y$$

- Plugging this in yields

$$\eta N(t) \frac{\beta L_Y}{r} = \beta \frac{Y(t)}{L_Y} \quad => \quad \eta N(t) \frac{\beta L_Y}{r} = \frac{\beta}{1-\beta} N(t)$$

- We can further simplify this expression to

$$r = (1-\beta)\eta L_Y$$

- We see from this that free entry into innovation yields a condition that links the interest rate (ultimately household patience) and the allocation of labor to production versus research

Market Clearing

- Goods market clearing implies:

$$C(t) + X(t) = Y(t)$$

where

$$X(t) = \int_0^{N(t)} \psi x(i,t) di$$

- Labor market clearing implies:

$$L_Y + L_R = L$$

Balanced Growth Path

- Consider again output of final goods

$$Y(t) = \frac{1}{1-\beta} N(t) L_Y$$

Since L_Y is constant, the growth rate of $N(t)$ must be the same as the growth rate of output

- Next consider

$$\dot{N}(t) = \eta N(t) L_R(t)$$

Rearranging this equation yields:

$$g = \frac{\dot{N}(t)}{N(t)} = \eta L_R$$

Balanced Growth Path

- We now have four equations in four unknown variables:

$$g = \frac{1}{\theta}(r - \rho) \qquad r = (1 - \beta)\eta L_Y$$

$$L_Y + L_R = L \qquad g = \eta L_R$$

- Solving these for g yields:

$$g = \frac{(1 - \beta)\eta L - \rho}{(1 - \beta) + \theta}$$

Balanced Growth Path

- To summarize:

$$g = \frac{(1-\beta)\eta L - \rho}{(1-\beta)+\theta}$$

- Intuitively: Growth is increasing in
  - Productivity of R&D (i.e., η)
  - Patience (i.e., falling in ρ)
  - Size of the population (i.e., L)
- The last of these is the scale effect we talked about last lecture

Is the Economy Pareto Optimal?

Two sources of market failure:

- Monopolistic competition: Prices are set at a markup over marginal cost and level of output is therefore too low
- Inefficient amount of innovation: Leads growth to be too low

Optimal Allocation

- We next solve for the optimal allocation
- Solution to the social planner problem of maximizing utility subject only to technological constraints
- Useful to do this in two steps:
  1. Optimal use of $x(i, t)$
  2. Optimal path for $C(t)$, $N(t)$, $L_Y(t)$

Optimal use of $x(i,t)$

- Goods market clearing can be written:

$$C(t) = Y(t) - X(t)$$
$$= \frac{1}{1-\beta} L_Y(t)^{\beta} \int_0^{N(t)} x(i,t)^{1-\beta} di - \int_0^{N(t)} \psi x(i,t) di$$

- The right-hand-side is "net output"

- The static optimum involves maximizing net output

- Differentiating net output with respect to $x(i,t)$ and setting the resulting expression equal to zero yields:

$$x^S(i,t) = (1-\beta)^{-1/\beta} L_Y^S(t)$$

where superscript S denotes "social planner solution"

Optimal Use of $x(i,t)$

- Market solution:

$$x(i,t) = L_Y(t)$$

- Social planner solution:

$$x^S(i,t) = (1-\beta)^{-1/\beta} L_Y^S(t)$$

- $x^S(i,t) > x(i,t)$ because social planner eliminates monopoly markup

OPTIMAL USE OF $x(i,t)$

- Plugging $x^S(i,t)$ into production function for final output yields

$$Y^S(t) = (1-\beta)^{-1/\beta} N^S(t) L_Y^S(t)$$

- And net output becomes

$$\begin{aligned} C^S(t) &= (1-\beta)^{-1/\beta} N^S(t) L_Y^S(t) - \int_0^{N(t)} \psi x^S(i,t) di \\ &= (1-\beta)^{-1/\beta} N^S(t) L_Y^S(t) - (1-\beta)^{-(1-\beta)/\beta} N^S(t) L_Y^S(t) \\ &= (1-\beta)^{-1/\beta} \beta N^S(t) L_Y^S(t) \end{aligned}$$

Optimal Path for $C(t)$, $N(t)$, $L_Y(t)$

- The social planner problem then becomes

$$\max \int_0^\infty \exp(-\rho t) \frac{C(t)^{1-\theta}}{1-\theta} dt$$

subject to

$$C(t) = (1-\beta)^{-1/\beta} \beta N(t) L_Y(t)$$

$$\dot{N}(t) = \eta N(t) L_R(t)$$

$$L_R(t) + L_Y(t) = L$$

OPTIMAL PATH FOR $C(t)$, $N(t)$, $L_Y(t)$

- We can simplify this to:

$$\max \int_0^\infty \exp(-\rho t)\frac{C(t)^{1-\theta}}{1-\theta}dt$$

subject to

$$\dot{N}(t) = \eta[N(t)L - (1-\beta)^{1/\beta}\beta^{-1}C(t)]$$

- We can now set up a current value Hamiltonian

$$\mathcal{H}(t) = \frac{C(t)^{1-\theta}}{1-\theta} + \mu(t)[\eta N(t)L - \eta(1-\beta)^{1/\beta}\beta^{-1}C(t)]$$

OPTIMAL PATH FOR $C(t)$, $N(t)$, $L_Y(t)$

$$\mathcal{H}(t) = \frac{C(t)^{1-\theta}}{1-\theta} + \mu(t)[\eta N(t)L - \eta(1-\beta)^{1/\beta}\beta^{-1}C(t)]$$

- Differentiating $\mathcal{H}(t)$ with respect to $C(t)$ and $N(t)$ yields:

$$\mathcal{H}_C(t) = C(t)^{-\theta} - \eta(1-\beta)^{1/\beta}\beta^{-1}\mu(t) = 0$$

$$\mathcal{H}_N(t) = \eta L\mu(t) = \rho\mu(t) - \dot{\mu}(t)$$

Optimal Path for $C(t)$, $N(t)$, $L_Y(t)$

- Manipulation of these equations yields:

$$\mu(t) = \eta^{-1}(1-\beta)^{-1/\beta}\beta C(t)^{-\theta}$$

$$\frac{\dot{\mu}(t)}{\mu(t)} = -[\eta L - \rho]$$

- Combining these yields:

$$\frac{\dot{C}^S(t)}{C^S(t)} = \frac{1}{\theta}[\eta L - \rho]$$

Optimal Growth

- The growth rate chosen by the social planner is

$$g^S = \frac{1}{\theta}[\eta L - \rho]$$

- The growth rate of the market economy with patents:

$$g = \frac{1}{\theta}((1-\beta)\eta L_Y - \rho)$$

- Since $L > (1-\beta)L_Y$ we have the

$$g^S > g$$

- The market economy with patents yield suboptimally low growth

Reasons for Suboptimal Growth

- **Appropriability**: Monopolist cannot appropriate full social value of its invention. Therefore innovates too little
- **R&D Externality**: Inventor doesn't take into account that new knowledge (higher $N(t)$) raises the productivity of future invention. Therefore innovates too little

- In addition, level of output is too low due to intermediate good monopolists setting prices above marginal cost

Public Policy Response

- Model already incorporates permanent (perfectly enforceable) patents
  - Real world has temporary, imperfectly enforceable patents
- Subsidies for research (e.g., NIH, NSF, NASA, DoD, DoE, etc.)
  - Challenges: How to direct funds. How to raise funds.
- Prizes and social recognition for innovators / researchers
- Subsidies for production of patented products:
  - Challenges: How large? What is price elasticity of demand?

Welfare vs. Growth

- Welfare and growth are not the same
- A policy that reduces monopoly distortions today (e.g., allows a new drug class to be sold more cheaply) will raise current well-being but lower growth (if future inventors expect the same)
- Whether this is good on net depends on:
  - How patient society is
    (how it trades off well-being of current versus future generations)
  - How important recent discoveries are for well being
    (think HIV/AIDS drugs / a cure for cancer / etc.)

Appendix

Derivation of Dixit-Stiglitz Price Index

- Notice that

$$\left(\frac{C}{c_i}\right)^{\frac{1}{\phi}} = \lambda p_i \quad \rightarrow \quad c_i = C(\lambda p_i)^{-\phi}$$

- Plug this into the budget constraint to get

$$Z = \int_0^N p_i C(\lambda p_i)^{-\phi} di$$

- Using the fact that $Z = PC$ (follows from definition of P) and rearranging yields:

$$P = \lambda^{-\phi} \int_0^N p_i^{1-\phi} di$$

- Notice that $\lambda = P^{-1}$ and rearrange to get

$$P = \left[\int_0^N p_i^{1-\phi} di\right]^{\frac{1}{1-\phi}}$$

Back