Consumption-Based Asset Pricing

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BIG ASSET PRICING QUESTIONS

- Why is the return on the stock market so high?
  (Relative to the "risk-free rate")
- Why is the stock market so volatile?
- What does this tell us about the risk and risk aversion?

Consumption-Based Asset Pricing

- Consumption-based asset pricing starts from the Consumption Euler equation:

$$U'(C_t) = E_t[\beta U'(C_{t+1}) R_{i,t+1}]$$

- Where does this equation come from?
  - Consume \$1 less today
  - Invest in asset i
  - Use proceeds to consume \$ R_{it+1} tomorrow
- Two perspectives:
  - Consumption Theory: Conditional on R_{it+1}, determine path for C_t
  - Asset Pricing: Conditional on path for C_t, determine R_{it+1}

Prices, Payoffs, and Returns

- Return is defined as payoff divided by price:

$$R_{i,t+1} = \frac{X_{i,t+1}}{P_{i,t}}$$

  where

  - $X_{i,t+1}$ is (state contingent) payoff from asset i in period $t+1$
  - $P_{i,t}$ is price of asset i at time t

Consumption-Based Asset Pricing

$$U'(C_t) = \beta E_t[U'(C_{t+1})R_{i,t+1}]$$

- A little manipulation yields:

$$1 = E_t\left[\frac{\beta U'(C_{t+1})}{U'(C_t)}R_{i,t+1}\right]$$

- and using $R_{i,t+1} = X_{i,t+1}/P_{i,t}$:

$$P_{i,t} = E_t\left[\beta\frac{U'(C_{t+1})}{U'(C_t)}X_{i,t+1}\right]$$

- Fundamental equation of consumption-based asset pricing

Stochastic Discount Factor

$$1 = E_t\left[\frac{\beta U'(C_{t+1})}{U'(C_t)} R_{i,t+1}\right] \qquad P_{i,t} = E_t\left[\beta \frac{U'(C_{t+1})}{U'(C_t)} X_{i,t+1}\right]$$

- Stochastic discount factor:

$$M_{t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)}$$

- Yields:

$$1 = E_t\left[M_{t+1} R_{i,t+1}\right] \qquad P_{i,t} = E_t\left[M_{t+1} X_{i,t+1}\right]$$

Stochastic Discount Factor

$$P_{i,t} = E_t\left[M_{t+1} X_{i,t+1}\right]$$

- Stochastic discount factor generalizes standard notion of discount factor
- With no uncertainty, standard present value formula gives

$$P_{i,t} = \frac{1}{R_{f,t}} X_{i,t+1}$$

- Since gross interest rates are usually above one, the payoff sells "at a discount"
- In this case, $1/R_{f,t}$ is the discount factor
- M_{t+1} is the appropriate discount factor when there is risk

Stochastic Discount Factor

$$1 = E_t\left[M_{t+1} R_{i,t+1}\right] \qquad P_{i,t} = E_t\left[M_{t+1} X_{i,t+1}\right]$$

- Stochastic discount factor prices all assets!!
- All risk compensation for any asset can be incorporated by defining a single (random) variable M_{t+1} to discount payoffs with
- This (conceptually) simple view holds under the rather strong assumption that there exists a complete set of competitive markets

(Sometimes also called: pricing kernel, marginal rate of substitution, change of measure, or state-price density)

Multi-Period Assets

- Assets can have payoffs in multiple periods:

$$P_{i,t} = E_t[M_{t+1}(D_{i,t+1} + P_{i,t+1})]$$

  where $D_{i,t+1}$ is the dividend, and $P_{i,t+1}$ is (ex dividend) price

- Works for stocks, bonds, options, everything.

State-Prices and Arbitrage

- Suppose $P_{s,t,t+1}$ is the price at time t of the Arrow security that pays \$1 if state s occurs at time $t+1$ and zero otherwise
- Asset with payoffs X_{t+1} over states at time $t+1$ can be replicated with a bunch of Arrow securities
  - $X_{1,t+1}$ units of the Arrow security that pays off in state 1
  - $X_{2,t+1}$ units of the Arrow security that pays off in state 2
  - Etc.
- If asset markets are perfectly competitive, the price of asset with payoff X_{t+1} should be the same as the price of the collection of Arrow assets that yield the same payoff
- If not, then there would exist an **arbitrage opportunity** (i.e., opportunity for risk-less gain)

State-Prices and Arbitrage

- The price of any security can by written two ways:

$$P_{i,t} = \sum_s P_{s,t,t+1} X_{s,t+1}, \qquad P_{i,t} = E_t[M_{t+1} X_{t+1}]$$

  which implies

$$M_{s,t+1} = \frac{P_{s,t,t+1}}{\pi_{s,t+1}}$$

  where $\pi_{s,t+1}$ is the probability of state s in period $t+1$.

- This is why you sometimes see $E_t[M_{t+1} X_{t+1}]$ type terms in budget constraints

MODIGLIANI-MILLER THEOREM

- Suppose:
  - Markets are complete and perfectly competitive
    (no bankruptcy costs, no agency costs, etc.)
  - No taxes
- Then:
  - Capital structure of a firm doesn't matter for its value!
  - Dividend policy of a firm doesn't matter for its value!
  - Whether a firm buys insurance (hedges a risk) doesn't matter!

Modigliani-Miller Theorem

- Why does Modigliani-Miller theorem hold?
- Value of an asset is the sum of its parts:

$$P_{i,t} = \sum_s P_{s,t,t+1} X_{s,t+1}$$

- Why? Arbitrage!
- Consequence: Doesn't matter how the asset is sliced up!
  (as long as the total payoff is not changed)
  - For example, doesn't matter how payoff of a firm is divided between equity and debt

Modigliani-Miller Theorem

- Does hedging a risk raise the value of a firm?
- Let's adopt vector notation:
  - S state of the world in the future
  - X_{t+1} is an $S \times 1$ vector of payoffs in these states
  - P_t is an $S \times 1$ vector of state prices
- Value of Firm A before hedging risk:

$$P_t^A = P_t \cdot X_{t+1}^A$$

where X_{t+1}^A denotes the payoffs of firm A over future states

Modigliani-Miller Theorem

- Consider some other cashflow X_{t+1}^B
- Price of that cashflow:

$$P_t^B = P_t \cdot X_{t+1}^B$$

- Suppose the firm were to purchase this cashflow
- At that point the firm's value would be the value of the combined cashflow minus the price of the cashflow:

$$P_t \cdot [X_{t+1}^A + X_{t+1}^B] - P_t^B = P_t \cdot X_{t+1}^A + P_t \cdot X_{t+1}^B - P_t^B = P_t^A + P_t^B - P_t^B = P_t^A$$

- True of any cashflow!! (Hedge, Bond, Dividend, etc.)
- Flows from the linearity of the pricing: By arbitrage, assets are worth the sum of their parts

Power Utility

- Now suppose that

$$U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma}$$

- This utility function is sometimes called CRRA utility for "constant relative risk aversion"
- Relative risk aversion:

$$-\frac{U''(C)C}{U'(C)} = \gamma$$

- Why do we think that this utility function is reasonable?

Power Utility

- Consider agent with CRRA utility and wealth W facing portfolio choice between risky and risk-free asset. Fraction allocated to risky asset is independent of wealth.

  (CARA utility: Dollar amount invested in risky asset is independent of wealth)

- This feature makes model consistent with stable interest rate and risk premia in the presence of long-run growth

Introspection about γ

- Consider the following gamble: I flip a coin and ...
  - If it comes up heads, I multiply your lifetime income by 1 million
  - If it comes up tails, I reduce your lifetime income by XX%

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- What about 20% reduction? If yes, CRRA $<$ 5
- What about 30% reduction? If yes, CRRA $<$ 3
- What about 50% reduction? If yes, CRRA $<$ 2

Introspection about γ

- What fraction of your lifetime wealth would you be willing to pay to avoid a 50/50 risk of gaining or losing a share α of your lifetime wealth
  - $\alpha = 0.10$
  - $\alpha = 0.30$

Table 2.1
Relative risk premium $\hat{\pi}$ associated to the risk of gaining or losing (α% of wealth)

RRA	$\alpha = 10\%$	$\alpha = 30\%$
$\gamma = 0.5$	0.3	2.3
$\gamma = 1$	0.5	4.6
$\gamma = 4$	2.0	16.0
$\gamma = 10$	4.4	24.4
$\gamma = 40$	8.4	28.7

Source: Gollier (2001)

Power Utility

$$U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma}$$

- With time separable power utility, γ is also the inverse of the intertemporal elasticity of substitution

$$\frac{d\log(C_{t+1}/C_t)}{d\log(P_{t+1}/P_t)} = \frac{d\log(C_{t+1}/C_t)}{d\log R_{ft}} = \frac{1}{\gamma}$$

- Only one parameter. So, it plays many roles.

  (Also governs strength of wealth effect on labor supply)

Power Utility

$$U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma}$$

- Implies:

$$U'(C_t) = C_t^{-\gamma}$$

$$M_{t+1} = \frac{\beta U'(C_{t+1})}{U'(C_t)} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

Risk Free Rate

- For risk-free bonds we have:

$$1 = E_t[M_{t+1} R_{f,t}] \;\; => \;\; 1 = E_t[M_{t+1}] R_{f,t} \;\; => \;\; R_{f,t} = \frac{1}{E_t M_{t+1}}$$

- Since risk free return is risk free, it is determined at time t
- With power utility

$$R_{f,t} = 1/E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right]$$

Risk Free Rate

- If X_{t+1} is log-normal, then

$$\log E_t X_{t+1} = E_t \log X_{t+1} + \frac{1}{2} \mathrm{Var}_t \log X_{t+1}$$

- If we assume consumption growth is log-normal, we get:

$$r_{f,t} = \delta + \gamma E_t[\Delta \log C_{t+1}] - \frac{\gamma^2}{2} \sigma_t^2(\Delta \log C_{t+1})$$

where $\beta = e^{-\delta}$, $r_{f,t} = \log R_{f,t}$

- Risk-free rate is determined by
  - Discount rate δ
  - Expected consumption growth
  - Precautionary savings ($\frac{\gamma^2}{2} \sigma_t^2(\Delta \log C_{t+1})$)

Risk Adjustments

$$P_{i,t} = E_t\left[M_{t+1}X_{i,t+1}\right]$$

- The definition of covariance implies

$$\text{cov}_t(M_{t+1}, X_{i,t+1}) = E_t[M_{t+1}X_{i,t+1}] - E[M_{i,t+1}]E[X_{i,t+1}]$$

- Using this yields

$$P_{i,t} = E[M_{i,t+1}]E[X_{i,t+1}] + \text{cov}_t(M_{t+1}, X_{i,t+1})$$

- Using $R_{f,t} = 1/E_t[M_{t+1}]$ yields

$$P_{i,t} = \frac{E[X_{i,t+1}]}{R_{f,t}} + \text{cov}_t(M_{t+1}, X_{i,t+1})$$

RISK ADJUSTMENTS

$$P_{i,t} = \frac{E[X_{i,t+1}]}{R_{f,t}} + \text{cov}_t(M_{t+1}, X_{i,t+1})$$

- Second term is a risk adjustment
  - Price of asset is higher if payoff covaries positively with SDF
  - In this case, asset is a hedge
- With power utility:

$$P_{i,t} = \frac{E[X_{i,t+1}]}{R_{f,t}} + \beta \frac{\text{cov}_t(U'(C_{t+1}), X_{i,t+1})}{U'(C_t)}$$

- Asset is a hedge if:
  - Payoff covaries positively with marginal utility
  - Payoff covaries negatively with consumption

RISK ADJUSTED RETURNS

- Similar manipulations starting with $1 = E_t[M_{t+1} R_{i,t+1}]$ yield:

$$E_t[R_{i,t+1}] - R_{f,t} = -R_{f,t}\text{cov}_t(M_{t+1}, R_{i,t+1})$$

  and

$$E_t[R_{i,t+1}] - R_{f,t} = -\frac{\text{cov}_t(U'(C_{t+1}), R_{i,t+1})}{E_t[U'(C_{t+1})]}$$

- The return premium of asset *i* is higher if:
  - The return on asset *i* is negatively correlated with the M_{t+1}
  - The return on asset *i* is negatively correlated with the $U'(C_{t+1})$
  - The return on asset *i* is positively correlated with the C_{t+1}

Risk Adjustment with Power Utility

$$1 = E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{i,t+1} \right]$$

- Taking logs and assuming log-normality:

$$\begin{aligned} E_t r_{i,t+1} &= \delta + \gamma E_t[\Delta \log C_{t+1}] \\ &- \frac{1}{2}\sigma_t^2(\log R_{i,t+1}) - \frac{\gamma^2}{2}\sigma_t^2(\Delta \log C_{t+1}) + \gamma \text{cov}(\log R_{i,t+1}, \Delta \log C_{t+1}) \end{aligned}$$

- Combining this with expression for risk-free rate yields

$$E_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\sigma_t^2(\log R_{i,t+1}) = \gamma \text{cov}(\log R_{i,t+1}, \Delta \log C_{t+1})$$

Equity Premium

$$E_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\sigma_t^2(\log R_{i,t+1}) = \gamma \text{cov}(\log R_{i,t+1}, \Delta \log C_{t+1})$$

- Equity premium is risk aversion times covariance between consumption growth and return on equity
- But what is with this $\frac{1}{2}\sigma_t^2(\log R_{i,t+1})$ term?

Equity Premium

$$E_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\sigma_t^2(\log R_{i,t+1}) = \gamma \text{cov}(\log R_{i,t+1}, \Delta \log C_{t+1})$$

- Comes from difference between geometric and arithmetic returns:

$$\log E_t R_{i,t+1} - \log R_{f,t} = E_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\sigma_t^2(\log R_{i,t+1})$$

- Geometric mean: $E_t r_{i,t+1}$
- (Log of) Arithmetic mean: $\log E_t R_{i,t+1}$

$$\log E_t R_{i,t+1} = E_t r_{i,t+1} + \frac{1}{2}\sigma_t^2(\log R_{i,t+1})$$

- Standard deviation annual real return on stocks is roughly 18%

$$\frac{1}{2}\text{Var}_t \log R_{i,t+1} = \frac{1}{2}\sigma_i^2 = 1.5\%$$

Equity Premium

- Two ways to write equity premium equation:

$$E_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\sigma_t^2(\log R_{i,t+1}) = \gamma \text{cov}(\log R_{i,t+1}, \Delta \log C_{t+1})$$

$$\log E_t R_{i,t+1} - \log R_{f,t} = \gamma \text{cov}(\log R_{i,t+1}, \Delta \log C_{t+1})$$

- Also recall that log of expected gross return is approximately equal to the expected net return:

$$\log(1 + x) \approx x$$

for small x

MEHRA-PRESCOTT 85 ("TEXTBOOK VERSION")

- Complete markets
- Representative agent with CRRA preferences:

$$C_t^{-\gamma} = E_t[\beta C_{t+1}^{-\gamma} R_{i,t+1}]$$

- Endowment economy ("Lucas tree"):

$$\log C_{t+1} = \mu + \log C_t + \epsilon_{t+1}$$

$$\epsilon_{t+1} \sim \mathrm{N}(0, \sigma^2)$$

(Original consumption process is a little different from this.)

- Equity modeled as a claim to the consumption process :

$$R_{i,t+1} = R_{C,t+1}$$

Mehra-Prescott 85 ("textbook version")

- In this case, equity premium and risk-free rate:

$$\log E_t R_{C,t+1} - \log R_{f,t} = \gamma \text{var}_t(\Delta \log C_{t+1})$$

$$\log R_{f,t} = -\log \beta + \gamma E_t[\Delta \log C_{t+1}] - \frac{\gamma^2}{2} \text{var}_t(\Delta \log C_{t+1})$$

- Does this model fit the data?
- We need data on:
  - Average returns on stocks and risk-free asset
  - Mean and variance of consumption growth
- We need a view as to what values are "reasonable" for γ
  - Mehra-Prescott: Values of $\gamma < 10$ "admissible"

Table 1

Time periods	% growth rate of per capita real consumption		% real return on a relatively riskless security		% risk premium		% real return on S&P 500	
	Mean	Standard deviation	Mean	Standard deviation	Mean	Standard deviation	Mean	Standard deviation
1889–1978	1.83 (Std error = 0.38)	3.57	0.80 (Std error = 0.60)	5.67	6.18 (Std error = 1.76)	16.67	6.98 (Std error = 1.74)	16.54
1889–1898	2.30	4.90	5.80	3.23	1.78	11.57	7.58	10.02
1899–1908	2.55	5.31	2.62	2.59	5.08	16.86	7.71	17.21
1909–1918	0.44	3.07	−1.63	9.02	1.49	9.18	−0.14	12.81
1919–1928	3.00	3.97	4.30	6.61	14.64	15.94	18.94	16.18
1929–1938	−0.25	5.28	2.39	6.50	0.18	31.63	2.56	27.90
1939–1948	2.19	2.52	−5.82	4.05	8.89	14.23	3.07	14.67
1949–1958	1.48	1.00	−0.81	1.89	18.30	13.20	17.49	13.08
1959–1968	2.37	1.00	1.07	0.64	4.50	10.17	5.58	10.59
1969–1978	2.41	1.40	−0.72	2.06	0.75	11.64	0.03	13.11

Source: Mehra and Prescott (1985)

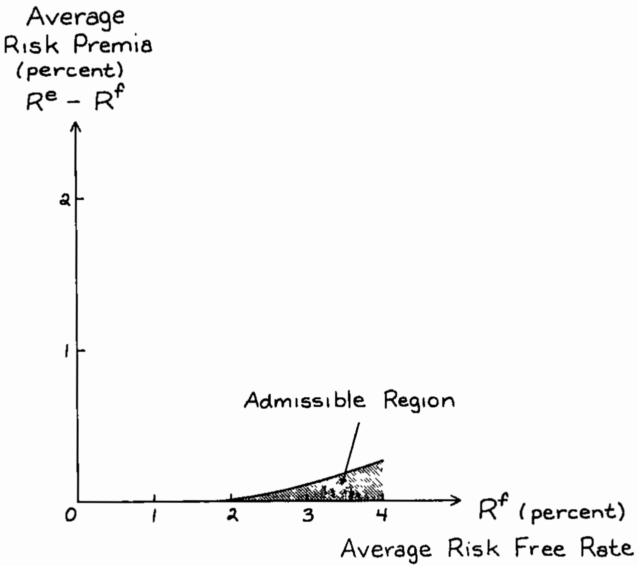


Fig. 4. Set of admissible average equity risk premia and real returns.

Source: Mehra and Prescott (1985). Values of equity premium and risk-free rate consistent with model given measured mean and variance of consumption growth and assuming $0 \leq \gamma \leq 10$ and $0 \leq \beta \leq 1$.

EQUITY PREMIUM

- Mehra-Prescott 85 made “extra” assumptions:
  - Endowment economy
  - Specific process for consumption growth
  - Equity is a consumption claim
- Equity premium equation can be evaluated independent of these assumptions:

$$E_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\sigma_t^2(\log R_{i,t+1}) = \gamma \text{cov}(\log R_{i,t+1}, \Delta \log C_{t+1})$$

Table 5
The equity premium puzzle[a]

Country	Sample period	$\overline{aer_e}$	$\sigma(er_e)$	$\sigma(m)$	$\sigma(\Delta c)$	$\rho(er_e, \Delta c)$	$\text{Cov}(er_e, \Delta c)$	RRA(1)	RRA(2)
USA	1947.2–1996.3	7.852	15.218	51.597	1.084	0.193	3.185	246.556	47.600
AUL	1970.1–1996.2	3.531	23.194	15.221	2.142	0.156	7.725	45.704	7.107
CAN	1970.1–1996.2	3.040	16.673	18.233	2.034	0.159	5.387	56.434	8.965
FR	1973.2–1996.2	7.122	22.844	31.175	2.130	–0.047	–2.295	< 0	14.634
GER	1978.4–1996.2	6.774	20.373	33.251	2.495	0.039	1.974	343.133	13.327
ITA	1971.2–1995.2	2.166	27.346	7.920	1.684	0.002	0.088	2465.323	4.703
JPN	1970.2–1996.2	6.831	21.603	31.621	2.353	0.100	5.093	134.118	13.440
NTH	1977.2–1996.1	9.943	15.632	63.607	2.654	0.023	0.946	1050.925	23.970
SWD	1970.1–1994.4	9.343	23.541	39.688	1.917	0.003	0.129	7215.176	20.705
SWT	1982.2–1996.2	12.393	20.466	60.553	2.261	–0.129	–5.978	< 0	26.785
UK	1970.1–1996.2	8.306	21.589	38.473	2.589	0.095	5.314	156.308	14.858
USA	1970.1–1996.3	5.817	16.995	34.228	0.919	0.248	3.875	150.136	37.255
SWD	1920–1993	6.000	18.906	31.737	2.862	0.169	9.141	65.642	11.091
UK	1919–1993	8.677	21.706	39.974	2.820	0.355	21.738	39.914	14.174
USA	1891–1994	6.258	18.534	33.767	3.257	0.497	30.001	20.861	10.366

[a] $\overline{aer_e}$ is the average excess log return on stock over a money market instrument, plus one half the variance of this excess return: $\overline{aer_e} = \overline{r_e - r_f} + \sigma^2(r_e - r_f)/2$. It is multiplied by 400 in quarterly data and 100 in annual data to express in annualized percentage points. $\sigma(er_e)$ and $\sigma(\Delta c)$ are the standard deviations of the excess log return $er_e = r_e - r_f$ and consumption growth Δc, respectively, multiplied by 200 in quarterly data and 100 in annual data to express in annualized percentage points. $\sigma(m) = 100\overline{aer_e}/\sigma(er_e)$ is calculated from equation (12) as a lower bound on the standard deviation of the log stochastic discount factor, expressed in annualized percentage points. $\rho(er_e, \Delta c)$ is the correlation of er_e and Δc. $\text{Cov}(er_e, \Delta c)$ is the product $\sigma(er_e)\sigma(\Delta c)\rho(er, \Delta c)$. RRA(1) is $100\overline{aer_e}/\text{Cov}(er_e, \Delta c)$, a measure of risk aversion calculated from equation (16) using the empirical covariance of excess stock returns with consumption growth. RRA(2) is $100\overline{aer_e}/\sigma(er_e)\sigma(\Delta c)$, a measure of risk aversion calculated using the empirical standard deviations of excess stock returns and consumption growth, but assuming perfect correlation between these series.
Abbreviations: AUL, Australia; CAN, Canada; FR, France; GER, Germany; ITA, Italy; JPN, Japan; NTH, Netherlands; SWD, Sweden; SWT, Switzerland; UK, United Kingdom; USA, United States of America.

Source: Campbell (1999)

Table 5 Long-Period Averages of Rates of Return						
Country	**Start**	**Stocks**	**Bills**	**Start**	**Bonds**	**Bills**
Part 1: OECD countries						
Australia	1876	0.1027 (0.1616)	0.0126 (0.0566)	1870	0.0352 (0.1157)	0.0125 (0.0569)
Belgium	--	--	--	1870	0.0291 (0.1584)**	0.0179 (0.1447)**
Canada	1916	0.0781 (0.1754)	--	1916	0.0392 (0.1199)	--
Denmark	1915	0.0750 (0.2300)	0.0265 (0.0652)	1870	0.0392 (0.1137)	0.0317 (0.0588)
Finland	1923	0.1268 (0.3155)	0.0128 (0.0935)	--	--	--
France	1870	0.0543 (0.2078)*	-0.0061 (0.0996)*	1870	0.0066 (0.1368)	-0.0079 (0.1000)
Germany	1870	0.0758 (0.2976)	-0.0153 (0.1788)	1924	0.0402 (0.1465)	0.0158 (0.1173)
Italy	1906	0.0510 (0.2760)	-0.0112 (0.1328)	1870	0.0173 (0.1879)	0.0046 (0.1191)
Japan	1894	0.0928 (0.3017)	-0.0052 (0.1370)	1883	0.0192 (0.1820)	0.0043 (0.1475)
Netherlands	1920	0.0901 (0.2116)**	0.0114 (0.0474)**	1881	0.0308 (0.1067)	0.0118 (0.0512)
New Zealand	1927	0.0762 (0.2226)	0.0234 (0.0529)	1926	0.0276 (0.1209)	0.0240 (0.0529)
Norway	1915	0.0716 (0.2842)	0.0098 (0.0782)	1877	0.0280 (0.1130)	0.0204 (0.0709)
Spain	1883	0.0610 (0.2075)†	0.0173 (0.0573)†	--	--	--
Sweden	1902	0.0923 (0.2347)	0.0180 (0.0719)	1922	0.0292 (0.0941)	0.0176 (0.0448)
Switzerland	1911	0.0726 (0.2107)††	0.0083 (0.0531)††	1916	0.0218 (0.0717)	0.0065 (0.0545)
U.K.	1870	0.0641 (0.1765)	0.0179 (0.0624)	1870	0.0280 (0.1049)	0.0179 (0.0624)
U.S.	1870	0.0827 (0.1866)	0.0199 (0.0482)	1870	0.0271 (0.0842)	0.0199 (0.0482)
Part 2: Non-OECD countries						
Chile	1895	0.1430 (0.4049)	-0.0094 (0.1776)	--	--	--
India	1921	0.0514 (0.2341)***	0.0133 (0.0835)***	1874	0.0191 (0.1147)	0.0240 (0.0785)
South Africa	1911	0.0890 (0.2006)	--	1911	0.0248 (0.1165)	--
Overall means†††	--	0.0814 (0.2449)	0.0085 (0.0880)	--	0.0266 (0.1234)	0.0147 (0.0805)

*missing 1940-41, **missing 1945-46, †missing 1936-40, ††missing 1914-16, ***missing 1926-27
†††Averages of means and standard deviations for 17 countries with stock and bill data and 15 countries with bond and bill data

Source: Barro and Ursua (2008)

Equity Premium

- Volatility of consumption seems to be relatively modest
- World seems to be a relatively safe place
- People must be very risk averse to not want to bid up prices of stocks
- High equity premium implies that stocks are cheap!!

Equity Premium is VERY BIG

- Suppose we invest \$ 1 in:
  - Equity with 8% real return
  - Tbills with 1% real return

Horizon	Equity	Tbills
1	1.08	1.01
5	1.50	1.05
10	2.15	1.10
25	6.85	1.28
50	46.90	1.64
100	2199.76	2.70

- Dutch (supposedly) bought Manhattan from natives for \$24 in 1626
- Suppose natives invested this in the stock market:

$$\$24 \times 1.08^{(2023-1626)} = \$4.46 \times 10^{14} = \$446 \text{ Trillion}$$

Equity Premium is VERY BIG ... Or Is It?

- Mean equity premium: $\approx 6.5\%$
- Standard deviation of equity premium: $\approx 18\%$
- Standard error on equity premium: $\sigma/\sqrt{T} = 2.1\%$ (post-WWII)
  $\sigma/\sqrt{T} = 1.5\%$ (post-1870)
- Using post-WWII standard error:
  - 95% confidence interval for equity premium: $[2.3\%, 10.7\%]$
- Perhaps last 100 years have been unusually good

GORDON GROWTH FORMULA

- What is the price of a dividend stream that growth at rate g and is discounted at rate r?

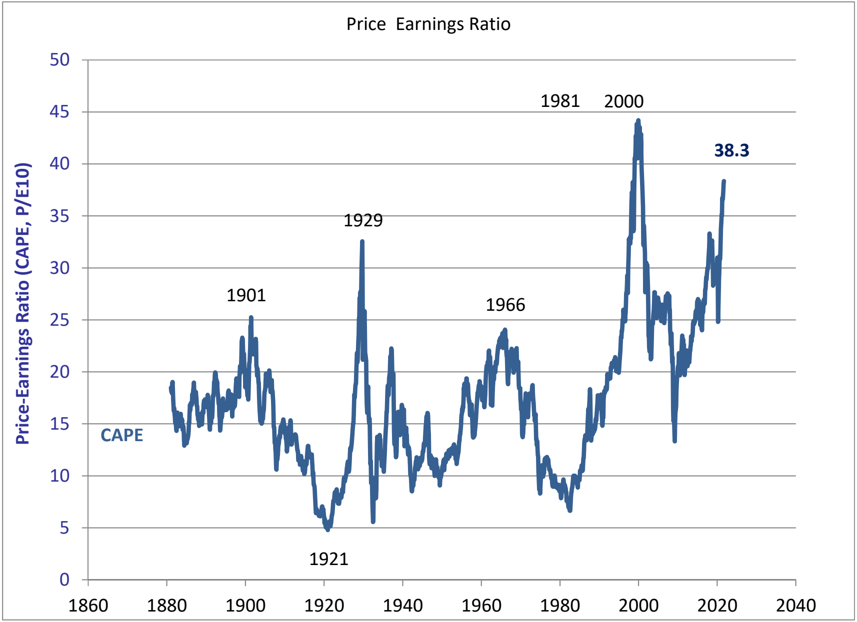
$$
\begin{aligned}
P_0 &= \frac{D_1}{1+r} + \frac{D_1(1+g)}{(1+r)^2} + \frac{D_1(1+g)^2}{(1+r)^3} + \ldots \\
&= \frac{D_1}{1+r}\left[1 + \left(\frac{1+g}{1+r}\right) + \left(\frac{1+g}{1+r}\right)^2 + \ldots\right] \\
&= \frac{D_1}{1+r}\left[\frac{1}{1-\frac{1+g}{1+r}}\right] \\
&= \frac{D_1}{1+r}\frac{1+r}{r-g}
\end{aligned}
$$

- Rearranging yields

$$\frac{P_0}{D_1} = \frac{1}{r-g}$$

Equity Premium is VERY BIG ... Or Is It?

- Relative to prior history, 20th century was good for growth and stocks
- Gordon growth formula:
$$\frac{P}{D} = \frac{1}{r - g}$$
- Maybe expectations about future growth have risen (i.e., $\uparrow g$)
- Maybe equity premium has fallen (i.e., $\downarrow r$)
- Would yield an unusually high return not to be repeated in the future



Source: Robert Shiller's website. Last updated in 2021.

Hansen-Jaganathan Bound

$$E_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\sigma_t^2(\log R_{i,t+1}) = -\text{cov}(\log R_{i,t+1}, \Delta \log M_{t+1})$$

- Let's adopt the notation: $E_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\sigma_i^2 = -\sigma_{im}$
- Definition of correlation coefficient:

$$\rho_{im} = \frac{\sigma_{im}}{\sigma_i \sigma_m}$$

$$-1 \leq \rho_{im}$$

$$\sigma_m \geq \frac{-\sigma_{im}}{\sigma_i}$$

$$\sigma_m \geq \frac{E_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\sigma_i^2}{\sigma_i}$$

- Ratio on right-hand-side called “Sharpe ratio”

HANSEN-JAGANATHAN BOUND

$$\sigma_m \geq \frac{E_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\sigma_i^2}{\sigma_i}$$

- Sharp ratio for stocks: 0.4
- Sharp ratio for other assets: >1
- Hansen-Jaganathan bound implies that volatility of stochastic discount factor is enormous
- Seems implausible

Risk-Free Rate Puzzle

$$\log R_{f,t} = \delta + \gamma E_t[\Delta \log C_{t+1}] - \frac{\gamma^2}{2}\text{var}_t(\Delta \log C_{t+1})$$

- $\text{var}_t(\Delta \log C_{t+1}) << E_t[\Delta \log C_{t+1}]$
- High value of γ therefore implies high risk free rate
- What is the intuition for this?

RISK-FREE RATE PUZZLE

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  - Consumers must be compensated a lot to allow their consumption profile to be upward sloping
  - This is γ acting in its incarnation as 1/IES

Risk-Free Rate Puzzle

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- High value of γ therefore implies high risk free rate
- What is the intuition for this?
  - Consumers must be compensated a lot to allow their consumption profile to be upward sloping
  - This is γ acting in its incarnation as 1/IES
- To get a low risk-free rate, $\beta > 1$

Table 6
The riskfree rate puzzle[a]

Country	Sample period	$\overline{r_f}$	$\overline{\Delta c}$	$\sigma(\Delta c)$	RRA(1)	TPR(1)	RRA(2)	TPR(2)
USA	1947.2–1996.3	0.794	1.908	1.084	246.556	–112.474	47.600	–76.710
AUL	1970.1–1996.2	1.820	1.854	2.142	45.704	–34.995	7.107	–10.196
CAN	1970.1–1996.2	2.738	1.948	2.034	56.434	–41.346	8.965	–13.066
FR	1973.2–1996.2	2.736	1.581	2.130	< 0	N/A	14.634	–15.536
GER	1978.4–1996.2	3.338	1.576	2.495	343.133	>1000	13.327	–12.142
ITA	1971.2–1995.2	2.064	2.424	1.684	>1000	>1000	4.703	–9.021
JPN	1970.2–1996.2	1.538	3.416	2.353	134.118	41.222	13.440	–39.375
NTH	1977.2–1996.1	3.705	1.466	2.654	>1000	>1000	23.970	–11.201
SWD	1970.1–1994.4	1.520	0.750	1.917	>1000	>1000	20.705	–6.126
SWT	1982.2–1996.2	1.466	0.414	2.261	< 0	N/A	26.785	8.698
UK	1970.1–1996.2	1.081	2.025	2.589	156.308	503.692	14.858	–21.600
USA	1970.1–1996.3	1.350	1.710	0.919	150.136	–160.275	37.255	–56.505
SWD	1920–1993	2.073	1.748	2.862	65.642	63.778	11.091	–12.274
UK	1919–1993	1.198	1.358	2.820	39.914	10.364	14.174	–10.057
USA	1891–1994	1.955	1.742	3.257	20.861	–11.305	10.366	–10.406

[a] $\overline{r_f}$ is the mean money market return from Table 2, in annualized percentage points. $\overline{\Delta c}$ and $\sigma(\Delta c)$ are the mean and standard deviation of consumption growth from Table 3, in annualized percentage points. RRA(1) and RRA(2) are the risk aversion coefficients from Table 5. $TPR(1) = \overline{r_f} - RRA(1)\overline{\Delta c} + RRA(1)^2\sigma^2(\Delta c)/200$, and $TPR(2) = \overline{r_f} - RRA(2)\overline{\Delta c} + RRA(2)^2\sigma^2(\Delta c)/200$. From Equation (17), these time preference rates give the real interest rate, in annualized percentage points, that would prevail if consumption growth had zero mean and zero standard deviation and risk aversion were RRA(1) or RRA(2), respectively.

Is the Equity Premium a Liquidity Premium?

- Perhaps low return on short term bonds is a liquidity premium for "money-like" features
- Campbell argues against this based on the term premium:
  - Long-term bonds don't have this type of liquidity premium
  - Yet their returns are only slightly higher than those of short-term bonds

Table 7
International yield spreads and bond excess returns[a]

Country	Sample period	$\bar{s}$	$\sigma(s)$	$\rho(s)$	$\overline{er_b}$	$\sigma(er_b)$	$\rho(er_b)$
USA	1947.2–1996.4	1.199	0.999	0.783	0.011	8.923	0.070
AUL	1970.1–1996.3	0.938	1.669	0.750	0.156	8.602	0.162
CAN	1970.1–1996.3	1.057	1.651	0.819	0.950	9.334	–0.009
FR	1973.2–1996.3	0.917	1.547	0.733	1.440	8.158	0.298
GER	1978.4–1996.3	0.991	1.502	0.869	0.899	7.434	0.117
ITA	1971.2–1995.3	–0.200	2.025	0.759	–1.386	9.493	0.335
JPN	1970.2–1996.3	0.593	1.488	0.843	1.687	9.165	–0.058
NTH	1977.2–1996.2	1.212	1.789	0.574	1.549	7.996	0.032
SWD	1970.1–1995.1	0.930	2.046	0.724	–0.212	7.575	0.244
SWT	1982.2–1996.3	0.471	1.655	0.755	1.071	6.572	0.268
UK	1970.1–1996.3	1.202	2.106	0.893	0.959	11.611	–0.057
USA	1970.1–1996.4	1.562	1.190	0.737	1.504	10.703	0.033
SWD	1920–1994	0.284	1.140	0.280	–0.075	6.974	–0.185
UK	1919–1994	1.272	1.505	0.694	0.318	8.812	–0.098
USA	1891–1995	0.720	1.550	0.592	0.172	6.499	0.153

[a] $\bar{s}$ is the mean of the log yield spread, the difference between the log yield on long-term bonds and the log 3-month money market return, expressed in annualized percentage points. $\sigma(s)$ is the standard deviation of the log yield spread and $\rho(s)$ is its first-order autocorrelation. $\overline{er_b}$, $\sigma(er_b)$, and $\rho(er_b)$ are defined in the same way for the excess 3-month return on long-term bonds over money market instruments, where the bond return is calculated from the bond yield using the par-bond approximation given in Campbell, Lo and MacKinlay (1997), Chapter 10, equation (10.1.19). Full details of this calculation are given in the Data Appendix.
Abbreviations: AUL, Australia; CAN, Canada; FR, France; GER, Germany; ITA, Italy; JPN, Japan; NTH, Netherlands; SWD, Sweden; SWT, Switzerland; UK, United Kingdom; USA, United States of America.

Equity Premium + Risk-Free Rate Puzzles

Restatement of Problem:

- To fit equity premium evidence, need high risk aversion
- High risk aversion implies low IES (with CRRA utility)
- Low IES implies high risk-free interest rate

"Obvious" solution:

- Consider preferences where IES may differ from 1/CRRA
- Make IES **and** CRRA high

- Epstein-Zin-Weil preferences deliver this

Epstein-Zin-Weil Preferences

- Epstein-Zin (1989, 1991) and Weil (1989) propose:

$$U_t = \left\{ (1-\delta) C_t^{\frac{1-\gamma}{\theta}} + \delta \left(E_t U_{t+1}^{1-\gamma} \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}}$$

- Parameters:

$$\theta = \frac{1-\gamma}{1-1/\psi}$$

  - γ: Coefficient of relative risk aversion
  - ψ: Intertemporal elasticity of substitution
- Falls outside expected utility framework
- Large literature about “weird” properties

- Consumption Euler equation with Epstein-Zin-Weil preferences:

$$1 = E_t \left[\beta^\theta \left(\frac{C_{t+1}}{C_t} \right)^{-\theta/\psi} (1 + R_{W,t+1})^{-(1-\theta)} (1 + R_{i,t+1}) \right]$$

- $R_{W,t+1}$ return on wealth

CRRA or IES??

- With power utility case, it is not clear whether γ appears in a particular equation because it is the CRRA or because it is 1/IES
- This is clarified in EZW case:

$$E_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\sigma_i^2 = \theta \frac{\sigma_{ic}}{\psi} + (1-\theta)\sigma_{iw}$$

$$r_{f,t} = -\log\beta + \frac{1}{\psi} E_t \Delta \log C_{t+1} + \frac{1}{2}(\theta - 1)\sigma_w^2 - \frac{1}{2}\frac{\theta}{\psi^2}\sigma_c^2$$

- Since both γ and ψ can be big at the same time,
  EP and RF puzzles can be resolved
- But are large values of γ and ψ “reasonable”